www.ietdl.org

Published in IET Control Theory and Applications
Received on 4th September 2013
Revised on 9th May 2014
Accepted on 15th May 2014
doi: 10.1049/iet-cta.2014.0293

ISSN 1751-8644

Robust extended Kalman filter for attitude estimation with multiplicative noises and unknown external disturbances

Qian Hua-ming[1], *Huang Wei*[1], *Qian Lin-chen*[2], *Shen Chen*[1]

[1] *College of Automation, Harbin Engineering University, 150001 Harbin, People's Republic of China*
[2] *College of Computer Science and Engineering, Beijing University of Aeronautics and Astronautics, Beijing 102206, People's Republic of China*
E-mail: huangwei2393@163.com

Abstract: This study is concerned with the robust extended Kalman filtering problem for non-linear attitude estimation systems with multiplicative noises and unknown external disturbances. The multiplicative noises are modelled by random variables with bounded variance. The unknown external disturbances are described to lie in bounded set. The objective of the addressed attitude estimation problem is to design a filter such that, in the presence of both the multiplicative noises and unknown external disturbances, an optimised upper bound on the state estimation error variance can be guaranteed. Thus, a robust extended Kalman filter (REKF) is presented for attitude estimation with multiplicative noises and unknown external disturbances. Compared with the traditional extended Kalman filter in attitude estimation, the proposed algorithm takes into consideration the effects of multiplicative noises and unknown external disturbances. Moreover, the stability of the proposed REKF can be proved under certain conditions by utilising the stochastic stability theory. Finally, the simulation results demonstrate the effectiveness of the proposed REKF.

1 Introduction

The attitude estimation system consisting of the gyroscopes and the star sensors is widely applicable for the aircraft and satellite because of the high measurement precision. To its quaternion non-linear attitude model, many attitude estimation filtering approaches based on extended Kalman filter (EKF) have been developed [1–5]. However, the EKF is only suitable for the accurate and known model with additive noises. In the case that there are model uncertainties in the system model, the performance of the EKF can be seriously affected. Hence, in the past few decades, many research attentions have been attracted to focus on the non-linear filtering problem with model uncertainties, including the H_∞ filter [6–8], set-valued non-linear filter [9–11], mixed H_2/H_∞ filter [12, 13] and robust filter design [14–18]. Among them, the robust filter design has been investigated to be a valid way for dealing with the non-linear system estimation problem with model uncertainties, but most of the results have been only concerned with a single model uncertainty and additive noises. However, the treatments of the important class of noises, namely multiplicative noises (also called state-dependent noises), play an important role in the actual engineering applications. For the attitude estimation system, the quaternion process noises are the multiplicative noises. The literatures [3, 19] have discussed the state-dependent noises in detail and gave a simplified expression for the covariance matrices of the state-dependent noises. In this paper, the multiplicative noises can be viewed as a model uncertainty. Recently, the non-linear filtering problem with multiplicative noises can be solved by employing a recursive design in the literature [20], but the situations that another model uncertainty exists in the system are not taken into consideration.

Apart from the multiplicative noises, unknown external disturbances are inevitably occurring in the attitude estimation systems because of the influence of the vibration of the carrier during running, which leads to the measurement error [21]. Although the measurement error may degrade the performance of the system, and result in the inconsistency of the output attitude information, very little researches have been made to deal with the filtering problem with unknown external disturbances in the attitude estimation. It is worth mentioning that the unknown external disturbances are assumed to appear in a prior way. However, in practical applications, the assumption is not always existence. Different from [21], in that the unknown external disturbances are only seen as random variables with bounded variance, a typical way is to make unknown external disturbances lie in bounded set in this paper. Up to now, to the best of authors' knowledge, the robust attitude estimation filtering problem with the multiplicative noises and unknown external disturbances has not been reported. Therefore an explicit and systematic solution to this question will need to be developed.

Motivated by the above-mentioned issue, this paper proposes to design the EKF for attitude estimation system in the case that there are multiplicative noises and unknown external disturbances. The considered multiplicative noises are expressed by zero mean Gaussian noises. The description of the unknown external disturbances is to lie in bounded set. A recursive robust EKF (REKF) is developed to deal with the attitude estimation filtering problem with the multiplicative noises and unknown external disturbances. Based on the structure of the EKF, the proposed algorithm designs an optimal upper bound of the prediction error and the filtering error covariance matrices. In additive, the estimation error of the REKF is bounded in mean square under certain conditions. Simulation results show the efficiency of the proposed filter.

2 Uncertainty model for attitude estimation

2.1 Gyro model

Assume that the gyro sampling interval is Δt. A common sensor that measured angular rate is a rate-integrating gyro. For this sensor, a widely used discrete-time three-axis gyro model is given by the following equations [3]

$$\begin{cases} \tilde{\boldsymbol{\omega}}_k = \boldsymbol{\omega}_k + \boldsymbol{\beta}_k + \boldsymbol{\eta}_v \\ \boldsymbol{\beta}_{k+1} = \boldsymbol{\beta}_k + \boldsymbol{\eta}_u \end{cases} \tag{1}$$

where, $\tilde{\boldsymbol{\omega}}_k$ is the gyro measured angular rate at time k; $\boldsymbol{\beta}_k$ is the gyro bias at time k; $\boldsymbol{\omega}_k$ is the true angular rate at time k; $\boldsymbol{\eta}_v$ and $\boldsymbol{\eta}_u$ are independent Gaussian white-noise processes with zero means and covariances σ_v^2 and $\sigma_u^2\Delta t$.

2.2 Process model

According to Markley [22], the orientation kinematics is formulated as

$$\dot{\boldsymbol{q}} = \frac{1}{2}\begin{bmatrix} \boldsymbol{\omega} \\ 0 \end{bmatrix} \otimes \boldsymbol{q} = \frac{1}{2}\Omega(\boldsymbol{\omega}) \cdot \boldsymbol{q} \tag{2}$$

where $\boldsymbol{q} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{\rho}^{\mathrm{T}} & q_4 \end{bmatrix}^{\mathrm{T}}$ is the attitude quaternion, which satisfies a normalisation constraint given by $\boldsymbol{q}^T\boldsymbol{q} = 1$; ρ is the vector component of quaternion; q_4 is the scalar part of quaternion; $\otimes$ is the quaternion product; $\boldsymbol{\omega} = [\omega_1 \ \omega_2 \ \omega_3]^{\mathrm{T}}$ is the angular rate vector of three-component and $\Omega(\boldsymbol{\omega}) = \begin{bmatrix} -[\boldsymbol{\omega}\times] & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^{\mathrm{T}} & 0 \end{bmatrix}$; $[\omega\times]$ is a cross-product matrix defined by $[\boldsymbol{\omega}\times] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$.

Then, the state vector consisting of the attitude quaternion $\boldsymbol{q}_k$ and the gyro bias β_k is defined as $\boldsymbol{x}_k = \begin{bmatrix} \boldsymbol{q}_k^{\mathrm{T}} & \boldsymbol{\beta}_k^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$. An exact state non-linear process model can be obtained as [3, 19] (see (3))

where (see equation at the bottom of the page)

and it is the additive noise with covariance $\boldsymbol{Q}_k = E(\boldsymbol{w}_k\boldsymbol{w}_k^{\mathrm{T}}) = \begin{bmatrix} \boldsymbol{0}_{4\times 4} & \boldsymbol{0}_{4\times 3} \\ \boldsymbol{0}_{3\times 4} & \Delta t\sigma_u^2\boldsymbol{I}_{3\times 3} \end{bmatrix}$; $s = 3$; η_{ik} is the zero mean multiplicative noise with covariance 1; $\boldsymbol{A}_{ik}$ are known scaling matrices with appropriate dimension, which can be expressed as

$$\boldsymbol{A}_{ik} = -\frac{\Delta t\sigma_v}{2}\begin{bmatrix} \boldsymbol{A}_{ik}^1 & \boldsymbol{0}_{4\times 3} \\ \boldsymbol{0}_{3\times 4} & \boldsymbol{0}_{3\times 3} \end{bmatrix}$$

where $\boldsymbol{A}_{1k}^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$, $\boldsymbol{A}_{2k}^1 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$, $\boldsymbol{A}_{3k}^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$

2.3 Measurement model

The star sensors can be modelled as

$$\boldsymbol{z}_k^i = \boldsymbol{A}(\boldsymbol{q}_k)\vec{\boldsymbol{r}}^i + v_k^i \tag{4}$$

where $\boldsymbol{z}_k^i$ is the measurement vector; $\boldsymbol{A}(\boldsymbol{q}_k)$ is the real attitude matrix at time k; $\vec{\boldsymbol{r}}^i$ is the reference vector of the star sensors, $\boldsymbol{v}_k^i$ is a zero mean Gaussian white-noise with covariance matrix $\sigma_s^2\boldsymbol{I}_{3\times 3}$; i is the corresponding number for different reference vectors.

$$\begin{aligned} \boldsymbol{x}_{k+1} = \begin{bmatrix} \boldsymbol{q}_{k+1} \\ \boldsymbol{\beta}_{k+1} \end{bmatrix} &= \begin{bmatrix} \cos\left(\dfrac{\|\tilde{\boldsymbol{\omega}}_k - \boldsymbol{\beta}_k\|\Delta t}{2}\right)\boldsymbol{I}_{4\times 4} + \sin\left(\dfrac{\|\tilde{\boldsymbol{\omega}}_k - \boldsymbol{\beta}_k\|\Delta t}{2}\right)\dfrac{\Omega(\tilde{\boldsymbol{\omega}}_k - \boldsymbol{\beta}_k)}{\|\tilde{\boldsymbol{\omega}}_k - \boldsymbol{\beta}_k\|} & \boldsymbol{0}_{4\times 3} \\ 0_{3\times 4} & \boldsymbol{I}_{3\times 3} \end{bmatrix}\begin{bmatrix} \boldsymbol{q}_k \\ \boldsymbol{\beta}_k \end{bmatrix} \\ &\quad + \begin{bmatrix} -\dfrac{\Delta t}{2}\Omega(\boldsymbol{\eta}_v) & \boldsymbol{0}_{4\times 3} \\ \boldsymbol{0}_{3\times 4} & \boldsymbol{0}_{3\times 3} \end{bmatrix}\begin{bmatrix} \boldsymbol{q}_k \\ \boldsymbol{\beta}_k \end{bmatrix} + \begin{bmatrix} \boldsymbol{0}_{4\times 1} \\ \boldsymbol{\eta}_u \end{bmatrix} \\ &= f(\boldsymbol{x}_k, \tilde{\boldsymbol{\omega}}_k) + \sum_{i=1}^{s}\boldsymbol{A}_{ik}\boldsymbol{\eta}_{ik}\boldsymbol{x}_k + \boldsymbol{w}_k \end{aligned} \tag{3}$$

$$f(\boldsymbol{x}_k, \tilde{\boldsymbol{\omega}}_k) = \begin{bmatrix} \left(\cos\left(\dfrac{\|\tilde{\boldsymbol{\omega}}_k - \boldsymbol{\beta}_k\|\Delta t}{2}\right)\boldsymbol{I}_{4\times 4} + \sin\left(\dfrac{\|\tilde{\boldsymbol{\omega}}_k - \boldsymbol{\beta}_k\|\Delta t}{2}\right)\dfrac{\Omega(\tilde{\boldsymbol{\omega}}_k - \boldsymbol{\beta}_k)}{\|\tilde{\boldsymbol{\omega}}_k - \boldsymbol{\beta}_k\|}\right)\boldsymbol{q}_k \\ \boldsymbol{\beta}_k \end{bmatrix}; \boldsymbol{w}_k = \begin{bmatrix} \boldsymbol{0}_{4\times 1} \\ \boldsymbol{\eta}_u \end{bmatrix}$$

doi: 10.1049/iet-cta.2014.0293

According to the definition of the quaternion in (2), the attitude matrix is related to the quaternion by

$$A(\boldsymbol{q}) = (q_4^2 - \boldsymbol{\rho}^{\mathrm{T}}\boldsymbol{\rho})\boldsymbol{I}_{3\times 3} + 2\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}} - 2q_4[\boldsymbol{\rho}\times] \tag{5}$$

As is well known, the carrier, such as aircraft or satellite, is inevitably influenced by the complicated external environment during running, which can lead to the occurrence of vibration or jitter, so that the unknown random disturbance is introduced into the star sensor measurement model. The measurement model with unknown external disturbances can be expressed as

$$z_k^i = (\boldsymbol{I}_{3\times 3} - [\boldsymbol{\varphi}_i\times])\boldsymbol{A}(\boldsymbol{q}_k)\vec{\boldsymbol{r}}^i + \boldsymbol{v}_k^i \tag{6}$$

where $\boldsymbol{\varphi}_i = [\varphi_{ix} \quad \varphi_{iy} \quad \varphi_{iz}]^{\mathrm{T}}$ is the unknown disturbance error vectors, which can be written as

$$-[\boldsymbol{\varphi}_i\times] = -\begin{bmatrix} 0 & -\varphi_{iz} & \varphi_{iy} \\ \varphi_{iz} & 0 & -\varphi_{ix} \\ -\varphi_{iy} & \varphi_{ix} & 0 \end{bmatrix} = \boldsymbol{M}_i\boldsymbol{\Delta}_i\boldsymbol{E}_i \tag{7}$$

where $\boldsymbol{M}_i = \begin{bmatrix} 0 & 0 & \sigma_{iy} & 0 & \sigma_{iz} & 0 \\ \sigma_{ix} & 0 & 0 & 0 & 0 & \sigma_{iz} \\ 0 & \sigma_{ix} & 0 & \sigma_{iy} & 0 & 0 \end{bmatrix}$; $\boldsymbol{\Delta}_i = \mathrm{diag}$ $([\Delta_{ix} \quad \Delta_{ix} \quad \Delta_{iy} \quad \Delta_{iy} \quad \Delta_{iz} \quad \Delta_{iz}])$

$$\boldsymbol{E}_i = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}; \quad \Delta_{ij} = \frac{\varphi_{ij}}{\sigma_{ij}}, \; j = x, y, z$$

The parameters σ_{ij} are positive constants. If σ_{ij} are large enough, the inequalities $\boldsymbol{\Delta}_i^{\mathrm{T}}\boldsymbol{\Delta}_i \le \boldsymbol{I}_{6\times 6}$ and $\boldsymbol{\Delta}_i\boldsymbol{\Delta}_i^{\mathrm{T}} \le \boldsymbol{I}_{6\times 6}$ can be satisfied [23]. To obtain the attitude information, three reference vectors are chosen. The non-linear measurement model is obtained as

$$\begin{aligned} z_k &= \begin{bmatrix} z_k^1 \\ z_k^2 \\ z_k^3 \end{bmatrix} = \begin{bmatrix} (\boldsymbol{I}_{3\times 3} - [\boldsymbol{\varphi}_1\times])\boldsymbol{A}(\boldsymbol{q}_k)\vec{\boldsymbol{r}}^1 \\ (\boldsymbol{I}_{3\times 3} - [\boldsymbol{\varphi}_2\times])\boldsymbol{A}(\boldsymbol{q}_k)\vec{\boldsymbol{r}}^2 \\ (\boldsymbol{I}_{3\times 3} - [\boldsymbol{\varphi}_3\times])\boldsymbol{A}(\boldsymbol{q}_k)\vec{\boldsymbol{r}}^3 \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_k^1 \\ \boldsymbol{v}_k^2 \\ \boldsymbol{v}_k^3 \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{A}(\boldsymbol{q}_k)\vec{\boldsymbol{r}}^1 \\ \boldsymbol{A}(\boldsymbol{q}_k)\vec{\boldsymbol{r}}^2 \\ \boldsymbol{A}(\boldsymbol{q}_k)\vec{\boldsymbol{r}}^3 \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_k^1 \\ \boldsymbol{v}_k^2 \\ \boldsymbol{v}_k^3 \end{bmatrix} \\ &\quad + \begin{bmatrix} \boldsymbol{M}_1\Delta_1 E_1 & \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 3} \\ \boldsymbol{0}_{3\times 3} & \boldsymbol{M}_2\boldsymbol{\Delta}_2\boldsymbol{E}_2 & \boldsymbol{0}_{3\times 3} \\ \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 3} & \boldsymbol{M}_3\boldsymbol{\Delta}_3\boldsymbol{E}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{A}(\boldsymbol{q}_k)\vec{\boldsymbol{r}}^1 \\ \boldsymbol{A}(\boldsymbol{q}_k)\vec{\boldsymbol{r}}^2 \\ \boldsymbol{A}(\boldsymbol{q}_k)\vec{\boldsymbol{r}}^3 \end{bmatrix} \\ &= h(x_k) + \boldsymbol{M}\boldsymbol{\Delta}\boldsymbol{E}h(x_k) + \boldsymbol{v}_k = h(\boldsymbol{x}_k) + \boldsymbol{M}\boldsymbol{\Delta}g(\boldsymbol{x}_k) + \boldsymbol{v}_k \end{aligned} \tag{8}$$

where $h(\boldsymbol{x}_k)$ is the non-linear function of the state and

$$h(x_k) = \begin{bmatrix} \boldsymbol{A}(\boldsymbol{q}_k)\vec{\boldsymbol{r}}^1 \\ \boldsymbol{A}(\boldsymbol{q}_k)\vec{\boldsymbol{r}}^2 \\ \boldsymbol{A}(\boldsymbol{q}_k)\vec{\boldsymbol{r}}^3 \end{bmatrix}; \quad \boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_1 & & \\ & \boldsymbol{M}_2 & \\ & & \boldsymbol{M}_3 \end{bmatrix}$$

$$\boldsymbol{E} = \begin{bmatrix} \boldsymbol{E}_1 & & \\ & \boldsymbol{E}_2 & \\ & & \boldsymbol{E}_3 \end{bmatrix}; \quad \boldsymbol{\Delta} = \begin{bmatrix} \boldsymbol{\Delta}_1 & & \\ & \boldsymbol{\Delta}_2 & \\ & & \boldsymbol{\Delta}_3 \end{bmatrix}$$

and $\boldsymbol{\Delta}\boldsymbol{\Delta}^T \le \boldsymbol{I}_{18\times 18}$; $g(\boldsymbol{x}_k) = \boldsymbol{E}h(\boldsymbol{x}_k)$; $\boldsymbol{v}_k$ is the zero mean Gaussian white-noise processes with covariance $\boldsymbol{R}_k = \sigma_s^2\boldsymbol{I}_{9\times 9}$.

3 Robust attitude estimation algorithm

3.1 Problem description

Consider the non-linear discrete-time dynamic system with multiplicative noises and unknown external disturbances

$$\begin{cases} \boldsymbol{x}_{k+1} = f(\boldsymbol{x}_k) + \boldsymbol{B}_k\boldsymbol{\eta}_k\phi(\boldsymbol{x}_k) + \sum_{i=1}^{s}\boldsymbol{A}_{ik}\boldsymbol{\eta}_{ik}\boldsymbol{x}_k + \boldsymbol{w}_k \\ z_k = h(\boldsymbol{x}_k) + \boldsymbol{D}_k\xi_k g(\boldsymbol{x}_k) + \sum_{i=1}^{r}\boldsymbol{C}_{ik}\boldsymbol{\xi}_{ik}\boldsymbol{x}_k + \boldsymbol{v}_k \end{cases} \tag{9}$$

where $\boldsymbol{x}_k \in \mathbf{R}^n$ is the state vector; $z_k \in \mathbf{R}^m$ is the measurement vector; $f(\boldsymbol{x}_k)$ and $h(\boldsymbol{x}_k)$ are non-linear process and measurement functions, respectively, $\boldsymbol{B}_k$, $\boldsymbol{A}_{ik}$, $\boldsymbol{D}_k$ and $\boldsymbol{C}_{ik}$ are known matrices with appropriate dimension; η_{ik} and ξ_{ik} are independent Gaussian white-noise with zero means and covariances 1; $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$ are independent process and measurement Gaussian noise with zero means and covariances $\boldsymbol{Q}_k$ and $\boldsymbol{R}_k$; η_k and ξ_k are unknown matrices satisfying the conditions $\boldsymbol{\eta}_k\boldsymbol{\eta}_k^{\mathrm{T}} \le \boldsymbol{I}_{p\times p}$ and $\xi_k\xi_k^{\mathrm{T}} \le \boldsymbol{I}_{q\times q}$; the non-linear functions $\phi(\boldsymbol{x}_k)$ and $g(\boldsymbol{x}_k)$ are known matrix functions, satisfying

$$\mathrm{E}[\phi(\boldsymbol{x}_k)\phi^{\mathrm{T}}(\boldsymbol{x}_k)] \le \Xi_k, \quad \mathrm{E}[g(\boldsymbol{x}_k)g^{\mathrm{T}}(\boldsymbol{x}_k)] \le \boldsymbol{\Theta}_k$$

where Ξ_k and Θ_k are known matrix.

For the uncertain system (9), the aim of the paper is to design a recursive estimation algorithm with the structure of the EKF, which makes the filtering prediction and estimation covariance have upper bounds on the existing of multiplicative noises and unknown external disturbances

$$\begin{aligned} \hat{\boldsymbol{x}}_{k+1|k} &= f(\hat{\boldsymbol{x}}_k) \\ \hat{\boldsymbol{x}}_{k+1} &= \hat{\boldsymbol{x}}_{k+1|k} + \boldsymbol{K}_{k+1}[z_{k+1} - h(\hat{\boldsymbol{x}}_{k+1|k})] \end{aligned} \tag{10}$$

where $\hat{\boldsymbol{x}}_k$ is the state estimate; $\hat{\boldsymbol{x}}_{k+1|k}$ is the one-step state prediction at time k; $\boldsymbol{K}_{k+1}$ is the gain parameter to be determined; The estimation error and its covariance matrix are defined as

$$\tilde{\boldsymbol{x}}_{k+1} = \boldsymbol{x}_{k+1} - \hat{\boldsymbol{x}}_{k+1} \tag{11}$$

$$\boldsymbol{P}_{k+1} = \mathrm{E}(\tilde{\boldsymbol{x}}_{k+1}\tilde{\boldsymbol{x}}_{k+1}^{\mathrm{T}}) \tag{12}$$

In the case that the uncertain model and the filtering structure (10) are given, the design problem is how to construct a sequence of positive-definite matrices $\Sigma_k (0 \le k \le n)$, which satisfies $\boldsymbol{P}_k \le \boldsymbol{\Sigma}_k$ and design the filtering gain $\boldsymbol{K}_k$ such that the upper bound of the estimation covariance Σ_k is minimised.

Remark 1: In real attitude estimation application, multiplicative noises result inevitably from the state non-linear process model of the system. In the literature [2, 3, 19], the covariance matrices of multiplicative noises are computed approximately by utilising the state estimation value. However, since there are the estimation errors, it is very different to calculate the covariance matrices of multiplicative noises accurately. From the state model (3), we can also see that multiplicative noises only relate to the real state, which leads to the unknown noise variance. Thus, as discussed in [20], it should be assumed as a model uncertainty. Besides, because of the spacecraft body librating and the influence of the external environment, unknown external disturbances are constantly encountered in the sensor measurement. Different from the assumption of the literature [21] that unknown external disturbances are Gaussian white noises with zero means and known covariances, unknown external disturbances are described to be norm-bounded. Hence, by considering the phenomenon of the multiplicative noises and unknown external disturbances, the new system model (9) is propitious to describe the actual situations in the attitude estimation system when the unknown external disturbances are occurring.

3.2 Error covariance matrix

The one-step state prediction estimation error and its corresponded covariance can be expressed as

$$\tilde{\boldsymbol{x}}_{k+1|k} = \boldsymbol{x}_{k+1} - \hat{\boldsymbol{x}}_{k+1|k} \tag{13}$$

$$\boldsymbol{P}_{k+1|k} = \mathrm{E}(\tilde{\boldsymbol{x}}_{k+1|k}\tilde{\boldsymbol{x}}_{k+1|k}^{\mathrm{T}}) \tag{14}$$

Substituting (9) and (10) into (13), the one-step prediction error can be rewritten as

$$\tilde{\boldsymbol{x}}_{k+1|k} = f(\boldsymbol{x}_k) - f(\hat{\boldsymbol{x}}_k) + \boldsymbol{B}_k\boldsymbol{\eta}_k\phi(\boldsymbol{x}_k) + \sum_{i=1}^{s}\boldsymbol{A}_{ik}\boldsymbol{\eta}_{ik}\boldsymbol{x}_k + \boldsymbol{w}_k \tag{15}$$

The non-linear functions $f(\boldsymbol{x}_k)$ can be linearised by utilising the Taylor series expansion around $\hat{\boldsymbol{x}}_k$

$$f(\boldsymbol{x}_k) = f(\hat{\boldsymbol{x}}_k) + \boldsymbol{F}_k\tilde{\boldsymbol{x}}_k + o(|\tilde{\boldsymbol{x}}_k|) \tag{16}$$

where $\boldsymbol{F}_k = \partial f(\boldsymbol{x}_k)/\partial \boldsymbol{x}_k|_{x_k=\hat{x}_k}$; $o(|\tilde{\boldsymbol{x}}_k|)$ represents the high-order terms of the Taylor series expansion.

According to the literatures [24], $o(|\tilde{\boldsymbol{x}}_k|)$ can be expressed as

$$o(|\tilde{\boldsymbol{x}}_k|) = \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k\tilde{\boldsymbol{x}}_k \tag{17}$$

where $\boldsymbol{A}_k \in \mathbf{R}^{n\times n}$ is a known problem-dependent scaling matrix; $\boldsymbol{L}_k \in \mathbf{R}^{n\times n}$ is a known tuning matrix; $\beta_k \in \mathbf{R}^{n\times n}$ is an unknown time-varying matrix accounting for the linearisation errors of the system model that satisfies $\boldsymbol{\beta}_k\boldsymbol{\beta}_k^{\mathrm{T}} \le \boldsymbol{I}_{n\times n}$; If the effect of the linearisation error can be neglected, $\boldsymbol{A}_k$ should be set to zero.

Substituting (16) and (17) into (15), the one-step prediction error is rewritten as

$$\tilde{\boldsymbol{x}}_{k+1|k} = (\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)\tilde{\boldsymbol{x}}_k + \boldsymbol{B}_k\boldsymbol{\eta}_k\phi(x_k) + \sum_{i=1}^{s}\boldsymbol{A}_{ik}\boldsymbol{\eta}_{ik}\boldsymbol{x}_k + \boldsymbol{w}_k \tag{18}$$

According to (14) and (18), the one-step prediction covariance can be expressed as

$$\begin{aligned}\boldsymbol{P}_{k+1|k} &= \mathrm{E}[(\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)\tilde{\boldsymbol{x}}_k\tilde{\boldsymbol{x}}_k^{\mathrm{T}}(\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)^{\mathrm{T}}] \\ &\quad + \mathrm{E}[\boldsymbol{B}_k\boldsymbol{\eta}_k\phi(\boldsymbol{x}_k)\phi^{\mathrm{T}}(\boldsymbol{x}_k)\boldsymbol{\eta}_k^{\mathrm{T}}\boldsymbol{B}_k^{\mathrm{T}}] \\ &\quad + \mathrm{E}[(\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)\tilde{\boldsymbol{x}}_k(\boldsymbol{B}_k\boldsymbol{\eta}_k\phi(\boldsymbol{x}_k))^{\mathrm{T}}] \\ &\quad + \mathrm{E}[\boldsymbol{B}_k\boldsymbol{\eta}_k\phi(\boldsymbol{x}_k)\tilde{\boldsymbol{x}}_k^{\mathrm{T}}(\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)^{\mathrm{T}}] \\ &\quad + \sum_{i=1}^{s}\boldsymbol{A}_{ik}\mathrm{E}(\boldsymbol{x}_k\boldsymbol{x}_k^{\mathrm{T}})\boldsymbol{A}_{ik}^{\mathrm{T}} + \boldsymbol{Q}_k + \boldsymbol{N}_1 + \boldsymbol{N}_1^{\mathrm{T}} + \boldsymbol{N}_2 \\ &\quad + \boldsymbol{N}_2^{\mathrm{T}} + \boldsymbol{N}_3 + \boldsymbol{N}_3^{\mathrm{T}} + \boldsymbol{N}_4 + \boldsymbol{N}_4^{\mathrm{T}} + \boldsymbol{N}_5 + \boldsymbol{N}_5^{\mathrm{T}}\end{aligned} \tag{19}$$

where

$$\begin{aligned}\boldsymbol{N}_1 &= (\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)\mathrm{E}(\tilde{\boldsymbol{x}}_k\boldsymbol{w}_k^{\mathrm{T}}) \\ \boldsymbol{N}_2 &= (\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)\mathrm{E}\left[\tilde{\boldsymbol{x}}_k\boldsymbol{x}_k^{\mathrm{T}}\left(\sum_{i=1}^{s}\boldsymbol{A}_{ik}\boldsymbol{\eta}_{ik}\right)^{\mathrm{T}}\right] \\ \boldsymbol{N}_3 &= \boldsymbol{B}_k\boldsymbol{\eta}_k\mathrm{E}\left[\phi(\boldsymbol{x}_k)\boldsymbol{x}_k^{\mathrm{T}}\left(\sum_{i=1}^{s}\boldsymbol{A}_{ik}\boldsymbol{\eta}_{ik}\right)^{\mathrm{T}}\right] \\ \boldsymbol{N}_4 &= \boldsymbol{B}_k\boldsymbol{\eta}_k\mathrm{E}[\phi(\boldsymbol{x}_k)\boldsymbol{w}_k^{\mathrm{T}}] \\ \boldsymbol{N}_5 &= \mathrm{E}\left[\sum_{i=1}^{s}\boldsymbol{A}_{ik}\boldsymbol{\eta}_{ik}\boldsymbol{x}_k\boldsymbol{w}_k^{\mathrm{T}}\right]\end{aligned} \tag{20}$$

Since $\boldsymbol{w}_k$ and η_{ik} are independent Gaussian white-noise with zero means, it is easy to know that the terms N_1, N_2, N_3, N_4 and N_5 are all equal to zero. Hence, we have

$$\begin{aligned}\boldsymbol{P}_{k+1|k} &= \mathrm{E}[(\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)\tilde{\boldsymbol{x}}_k\tilde{\boldsymbol{x}}_k^{\mathrm{T}}(\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)^{\mathrm{T}}] \\ &\quad + \mathrm{E}[\boldsymbol{B}_k\boldsymbol{\eta}_k\phi(\boldsymbol{x}_k)\phi^{\mathrm{T}}(\boldsymbol{x}_k)\boldsymbol{\eta}_k^{\mathrm{T}}\boldsymbol{B}_k^{\mathrm{T}}] \\ &\quad + \mathrm{E}[(\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)\tilde{\boldsymbol{x}}_k(\boldsymbol{B}_k\boldsymbol{\eta}_k\phi(\boldsymbol{x}_k))^{\mathrm{T}}] \\ &\quad + \mathrm{E}[\boldsymbol{B}_k\boldsymbol{\eta}_k\phi(\boldsymbol{x}_k)\tilde{\boldsymbol{x}}_k^{\mathrm{T}}(\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)^{\mathrm{T}}] \\ &\quad + \sum_{i=1}^{s}\boldsymbol{A}_{ik}\mathrm{E}(\boldsymbol{x}_k\boldsymbol{x}_k^{\mathrm{T}})\boldsymbol{A}_{ik}^{\mathrm{T}} + \boldsymbol{Q}_k\end{aligned} \tag{21}$$

The innovation of the filter is defined as

$$\tilde{\boldsymbol{z}}_{k+1} = \boldsymbol{z}_{k+1} - h(\hat{\boldsymbol{x}}_{k+1|k}) \tag{22}$$

Inserting (9) into (22), we have

$$\begin{aligned}\tilde{\boldsymbol{z}}_{k+1} &= h(\boldsymbol{x}_{k+1}) - h(\hat{\boldsymbol{x}}_{k+1|k}) + \boldsymbol{D}_{k+1}\boldsymbol{\xi}_{k+1}g(\boldsymbol{x}_{k+1}) \\ &\quad + \sum_{i=1}^{r}\boldsymbol{C}_{ik+1}\boldsymbol{\xi}_{ik+1}\boldsymbol{x}_{k+1} + \boldsymbol{v}_{k+1}\end{aligned} \tag{23}$$

The non-linear functions $h(\boldsymbol{x}_{k+1})$ can be linearised by utilising the Taylor series expansion around $\hat{\boldsymbol{x}}_{k+1|k}$

$$h(\boldsymbol{x}_{k+1}) = h(\hat{\boldsymbol{x}}_{k+1|k}) + \boldsymbol{H}_{k+1}\tilde{\boldsymbol{x}}_{k+1|k} + o(|\tilde{\boldsymbol{x}}_{k+1|k}|) \tag{24}$$

where $\boldsymbol{H}_{k+1} = \partial h(\boldsymbol{x}_{k+1})/\partial \boldsymbol{x}_{k+1}|_{x_{k+1}=\hat{x}_{k+1|k}}$; $o(|\tilde{\boldsymbol{x}}_{k+1|k}|)$ represents the high-order terms of the Taylor series expansion, which can be expressed as

$$o(|\tilde{\boldsymbol{x}}_{k+1|k}|) = \boldsymbol{C}_{k+1}\boldsymbol{\alpha}_{k+1}\boldsymbol{L}_{k+1}\tilde{\boldsymbol{x}}_{k+1|k} \tag{25}$$

where $\boldsymbol{C}_{k+1} \in \mathbf{R}^{m\times n}$ is a known problem-dependent scaling matrix; $\alpha_{k+1} \in \mathbf{R}^{n\times n}$ is an unknown time-varying matrix that satisfies $\boldsymbol{\alpha}_{k+1}\boldsymbol{\alpha}_{k+1}^{\mathrm{T}} \le \boldsymbol{I}_{n\times n}$.

Substituting (24) and (25) into (22), the innovation is given as

$$\begin{aligned}\tilde{\boldsymbol{z}}_{k+1} &= (\boldsymbol{H}_{k+1} + \boldsymbol{C}_{k+1}\boldsymbol{\alpha}_{k+1}\boldsymbol{L}_{k+1})\tilde{\boldsymbol{x}}_{k+1|k} + \boldsymbol{D}_{k+1}\boldsymbol{\xi}_{k+1}g(\boldsymbol{x}_{k+1}) \\ &\quad + \sum_{i=1}^{r}\boldsymbol{C}_{ik+1}\boldsymbol{\xi}_{ik+1}\boldsymbol{x}_{k+1} + \boldsymbol{v}_{k+1}\end{aligned} \tag{26}$$

IET Control Theory Appl., 2014, Vol. 8, Iss. 15, pp. 1523–1536
doi: 10.1049/iet-cta.2014.0293

From (10), (11) and (26), the estimation error can be obtained as (see (27))

Substituting (27) into (12), the filtering error covariance can be expressed as (see (28))

where

$$\begin{aligned}
\boldsymbol{G}_1 &= (\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{\alpha}_{k+1}\boldsymbol{L}_{k+1}) \\
&\quad \times \mathrm{E}\left[\tilde{\boldsymbol{x}}_{k+1|k}\boldsymbol{x}_{k+1}^{\mathrm{T}}\left(\sum_{i=1}^{r}\boldsymbol{C}_{ik+1}\boldsymbol{\xi}_{ik+1}\right)^{\mathrm{T}}\right]\boldsymbol{K}_{k+1}^{\mathrm{T}} \\
\boldsymbol{G}_2 &= (\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{\alpha}_{k+1}\boldsymbol{L}_{k+1}) \\
&\quad \times \mathrm{E}[\tilde{\boldsymbol{x}}_{k+1|k}\boldsymbol{v}_{k+1}^{\mathrm{T}}]\boldsymbol{K}_{k+1}^{\mathrm{T}} \\
\boldsymbol{G}_3 &= \boldsymbol{K}_{k+1}\boldsymbol{D}_{k+1}\boldsymbol{\xi}_{k+1} \\
&\quad \times \mathrm{E}\left[g(\boldsymbol{x}_{k+1})\boldsymbol{x}_{k+1}^{\mathrm{T}}\left(\sum_{i=1}^{r}\boldsymbol{C}_{ik+1}\boldsymbol{\xi}_{ik+1}\right)^{\mathrm{T}}\right]\boldsymbol{K}_{k+1}^{\mathrm{T}} \\
\boldsymbol{G}_4 &= \boldsymbol{K}_{k+1}\boldsymbol{D}_{k+1}\boldsymbol{\xi}_{k+1}\boldsymbol{E}[g(\boldsymbol{x}_{k+1})\boldsymbol{v}_{k+1}^{\mathrm{T}}]\boldsymbol{K}_{k+1}^{\mathrm{T}} \\
\boldsymbol{G}_5 &= \boldsymbol{K}_{k+1}\mathrm{E}\left[\sum_{i=1}^{r}\boldsymbol{C}_{ik+1}\boldsymbol{\xi}_{ik+1}\boldsymbol{x}_{k+1}\boldsymbol{v}_{k+1}^{\mathrm{T}}\right]\boldsymbol{K}_{k+1}^{\mathrm{T}} \qquad (29)
\end{aligned}$$

Similarly, because of the independent Gaussian white-noise ξ_{ik+1} and $\boldsymbol{v}_{k+1}$ with zero means, the corresponding terms $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, $\boldsymbol{G}_3$, $\boldsymbol{G}_4$ and $\boldsymbol{G}_5$ are also all equal to zero. Therefore the filtering error covariance can be reduced as (see (30))

Because of existing the model uncertainties and the linearisation errors, the matrices η_k, ξ_k, β_k and α_{k+1} are unknown, which makes that the one-step prediction covariance $\boldsymbol{P}_{k+1|k}$ and the filtering error covariance $\boldsymbol{P}_{k+1}$ from (21) and (30) cannot be obtained directly. In order to accomplish the design of the filter, an effective way is to calculate the upper bounds for the $\boldsymbol{P}_{k+1|k}$ and $\boldsymbol{P}_{k+1}$, and then design the filtering gain $\boldsymbol{K}_k$ to minimise the upper bounds.

3.3 Robust extended Kalman filter design

The derivation of the filter is based on the following lemma.

Lemma 1 [25]*:* Given matrices $\boldsymbol{A}$, $\boldsymbol{H}$, $\boldsymbol{E}$ and $\boldsymbol{F}$ with compatible dimensions such that $\boldsymbol{F}\boldsymbol{F}^{\mathrm{T}} \le \boldsymbol{I}$. Let $\boldsymbol{X}$ be a symmetric positive-definite matrix and γ be an arbitrary positive constant such that

$$\gamma^{-1}\boldsymbol{I} - \boldsymbol{E}\boldsymbol{X}\boldsymbol{E}^{\mathrm{T}} > 0$$

Then the following matrix inequality holds

$$\begin{aligned}
(\boldsymbol{A} + \boldsymbol{H}\boldsymbol{F}\boldsymbol{E})\boldsymbol{X}(\boldsymbol{A} + \boldsymbol{H}\boldsymbol{F}\boldsymbol{E})^{\mathrm{T}} &\le \boldsymbol{A}(\boldsymbol{X}^{-1} - \gamma\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E})^{-1}\boldsymbol{A}^{\mathrm{T}} \\
&\quad + \gamma^{-1}\boldsymbol{H}\boldsymbol{H}^{\mathrm{T}} \qquad (31)
\end{aligned}$$

Lemma 2 [26]*:* Let $\boldsymbol{A} = [a_{ij}]_{n\times n}$ be a real matrix and $\boldsymbol{B} = \mathrm{diag}(b_1, b_2, \ldots, b_n)$ be a diagonal random matrix. Then

$$\mathrm{E}\{\boldsymbol{B}\boldsymbol{A}\boldsymbol{B}^{\mathrm{T}}\} = \begin{bmatrix} \mathrm{E}\{b_1^2\} & \mathrm{E}\{b_1b_2\} & \cdots & \mathrm{E}\{b_1b_n\} \\ \mathrm{E}\{b_2b_1\} & \mathrm{E}\{b_2^2\} & \cdots & \mathrm{E}\{b_2b_n\} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{E}\{b_nb_1\} & \mathrm{E}\{b_nb_2\} & \cdots & \mathrm{E}\{b_n^2\} \end{bmatrix} \circ \boldsymbol{A}$$

where $\circ$ is the Hadamard product.

According to the lemma, the following theorem is given to construct the upper bounds for the $\boldsymbol{P}_{k+1|k}$ and $\boldsymbol{P}_{k+1}$.

$$\begin{aligned}
\tilde{\boldsymbol{x}}_{k+1} &= \tilde{\boldsymbol{x}}_{k+1|k} - \boldsymbol{K}_{k+1}\tilde{z}_{k+1} = (\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{\alpha}_{k+1}\boldsymbol{L}_{k+1})\tilde{\boldsymbol{x}}_{k+1|k} \\
&\quad - \boldsymbol{K}_{k+1}\left[\boldsymbol{D}_{k+1}\boldsymbol{\xi}_{k+1}g(\boldsymbol{x}_{k+1}) + \sum_{i=1}^{r}\boldsymbol{C}_{ik+1}\boldsymbol{\xi}_{ik+1}\boldsymbol{x}_{k+1} + \boldsymbol{v}_{k+1}\right] \qquad (27)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{P}_{k+1} &= \mathrm{E}[(\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\alpha_{k+1}\boldsymbol{L}_{k+1})\tilde{\boldsymbol{x}}_{k+1|k}\tilde{\boldsymbol{x}}_{k+1|k}^{\mathrm{T}}(\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\alpha_{k+1}\boldsymbol{L}_{k+1})^{\mathrm{T}}] \\
&\quad + \mathrm{E}[\boldsymbol{K}_{k+1}\boldsymbol{D}_{k+1}\boldsymbol{\xi}_{k+1}g(\boldsymbol{x}_{k+1})g^{\mathrm{T}}(\boldsymbol{x}_{k+1})\boldsymbol{\xi}_{k+1}^{\mathrm{T}}\boldsymbol{D}_{k+1}^{\mathrm{T}}\boldsymbol{K}_{k+1}^{\mathrm{T}}] + \boldsymbol{K}_{k+1}\sum_{i=1}^{r}\boldsymbol{C}_{ik+1}\mathrm{E}(\boldsymbol{x}_{k+1}\boldsymbol{x}_{k+1}^{\mathrm{T}})\boldsymbol{C}_{ik+1}^{\mathrm{T}}\boldsymbol{K}_{k+1}^{\mathrm{T}} + \boldsymbol{K}_{k+1}R_{k+1}\boldsymbol{K}_{k+1}^{\mathrm{T}} \\
&\quad - \mathrm{E}[(\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\alpha_{k+1}\boldsymbol{L}_{k+1})\tilde{\boldsymbol{x}}_{k+1|k}g^{\mathrm{T}}(\boldsymbol{x}_{k+1})\boldsymbol{\xi}_{k+1}^{\mathrm{T}}\boldsymbol{D}_{k+1}^{\mathrm{T}}\boldsymbol{K}_{k+1}^{\mathrm{T}}] \\
&\quad - \mathrm{E}[\boldsymbol{K}_{k+1}\boldsymbol{D}_{k+1}\boldsymbol{\xi}_{k+1}g(\boldsymbol{x}_{k+1})\tilde{\boldsymbol{x}}_{k+1|k}^{\mathrm{T}}(\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\alpha_{k+1}\boldsymbol{L}_{k+1})^{\mathrm{T}}] \\
&\quad - \boldsymbol{G}_1 - \boldsymbol{G}_1^{\mathrm{T}} - \boldsymbol{G}_2 - \boldsymbol{G}_2^{\mathrm{T}} + \boldsymbol{G}_3 + \boldsymbol{G}_3^{\mathrm{T}} + \boldsymbol{G}_4 + \boldsymbol{G}_4^{\mathrm{T}} + \boldsymbol{G}_5 + \boldsymbol{G}_5^{\mathrm{T}} \qquad (28)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{P}_{k+1} &= \mathrm{E}[(\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{\alpha}_{k+1}\boldsymbol{L}_{k+1})\tilde{\boldsymbol{x}}_{k+1|k}\tilde{\boldsymbol{x}}_{k+1|k}^{\mathrm{T}}(\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{\alpha}_{k+1}\boldsymbol{L}_{k+1})^{\mathrm{T}}] \\
&\quad + \mathrm{E}[\boldsymbol{K}_{k+1}\boldsymbol{D}_{k+1}\boldsymbol{\xi}_{k+1}g(\boldsymbol{x}_{k+1})g^{\mathrm{T}}(\boldsymbol{x}_{k+1})\boldsymbol{\xi}_{k+1}^{\mathrm{T}}\boldsymbol{D}_{k+1}^{\mathrm{T}}\boldsymbol{K}_{k+1}^{\mathrm{T}}] + \boldsymbol{K}_{k+1}\sum_{i=1}^{r}\boldsymbol{C}_{ik+1}\mathrm{E}(\boldsymbol{x}_{k+1}\boldsymbol{x}_{k+1}^{\mathrm{T}})\boldsymbol{C}_{ik+1}^{\mathrm{T}}\boldsymbol{K}_{k+1}^{\mathrm{T}} + \boldsymbol{K}_{k+1}\boldsymbol{R}_{k+1}\boldsymbol{K}_{k+1}^{\mathrm{T}} \\
&\quad - \mathrm{E}[(\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{\alpha}_{k+1}\boldsymbol{L}_{k+1})\tilde{\boldsymbol{x}}_{k+1|k}g^{\mathrm{T}}(\boldsymbol{x}_{k+1})\boldsymbol{\xi}_{k+1}^{\mathrm{T}}\boldsymbol{D}_{k+1}^{\mathrm{T}}\boldsymbol{K}_{k+1}^{\mathrm{T}}] \\
&\quad - \mathrm{E}[\boldsymbol{K}_{k+1}\boldsymbol{D}_{k+1}\boldsymbol{\xi}_{k+1}g(\boldsymbol{x}_{k+1})\tilde{\boldsymbol{x}}_{k+1|k}^{\mathrm{T}}(\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{\alpha}_{k+1}\boldsymbol{L}_{k+1})^{\mathrm{T}}] \qquad (30)
\end{aligned}$$

doi: 10.1049/iet-cta.2014.0293

Theorem 1: If there exist the positive scalars ε_1, ε_2, ε_3, ε_4, λ and μ, such that the following inequalities are satisfied

$$0.5\lambda^{-1}\boldsymbol{I}_{n\times n} - \boldsymbol{L}_k\boldsymbol{P}_k\boldsymbol{L}_k^{\mathrm{T}} > 0, \quad \mu^{-1}\boldsymbol{I}_{n\times n} - \boldsymbol{L}_{k+1}\boldsymbol{P}_{k+1|k}\boldsymbol{L}_{k+1}^{\mathrm{T}} > 0$$

then the following inequalities with respect to the $\boldsymbol{P}_{k+1|k}$ and $\boldsymbol{P}_{k+1}$ can be satisfied

$$\begin{aligned}\boldsymbol{P}_{k+1|k} \le{}& (1+\varepsilon_1)[\boldsymbol{F}_k(P_k^{-1} - 2\lambda\boldsymbol{L}_k^{\mathrm{T}}\boldsymbol{L}_k)^{-1}\boldsymbol{F}_k^{\mathrm{T}} + \lambda^{-1}\boldsymbol{A}_k\boldsymbol{A}_k^{\mathrm{T}}]\\ &+ (1+\varepsilon_1^{-1})\boldsymbol{B}_k\boldsymbol{\Xi}_k\boldsymbol{B}_k^{\mathrm{T}} + \boldsymbol{Q}_k\\ &+ \sum_{i=1}^{s}\boldsymbol{A}_{ik}[(1+\varepsilon_2)\boldsymbol{P}_k + (1+\varepsilon_2^{-1})\hat{\boldsymbol{x}}_k\hat{\boldsymbol{x}}_k^{\mathrm{T}}]\boldsymbol{A}_{ik}^{\mathrm{T}}\end{aligned} \tag{32}$$

$$\begin{aligned}\boldsymbol{P}_{k+1} \le{}& (1+\varepsilon_3)[(\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1})(\boldsymbol{P}_{k+1|k}^{-1} - \mu\boldsymbol{L}_{k+1}^{\mathrm{T}}\boldsymbol{L}_{k+1})^{-1}\\ &\times(\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1})^{\mathrm{T}} + \mu^{-1}\boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{C}_{k+1}^{\mathrm{T}}\boldsymbol{K}_{k+1}^{\mathrm{T}}]\\ &+ (1+\varepsilon_3^{-1})\boldsymbol{K}_{k+1}\boldsymbol{D}_{k+1}\boldsymbol{\Theta}_{k+1}\boldsymbol{D}_{k+1}^{\mathrm{T}}\boldsymbol{K}_{k+1}^{\mathrm{T}}\\ &+ \boldsymbol{K}_{k+1}\boldsymbol{R}_{k+1}\boldsymbol{K}_{k+1}^{\mathrm{T}} + \boldsymbol{K}_{k+1}\sum_{i=1}^{r}\boldsymbol{C}_{ik+1}[(1+\varepsilon_4)\boldsymbol{P}_{k+1|k}\\ &+ (1+\varepsilon_4^{-1})\hat{\boldsymbol{x}}_{k+1|k}\hat{\boldsymbol{x}}_{k+1|k}^{\mathrm{T}}]\boldsymbol{C}_{ik+1}^{\mathrm{T}}\boldsymbol{K}_{k+1}^{\mathrm{T}}\end{aligned} \tag{33}$$

Proof: Assume that ε_1 is a positive scalar. The following matrix inequality

$$\begin{aligned}&[\varepsilon_1^{1/2}(\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)\tilde{\boldsymbol{x}}_k - \varepsilon_1^{-1/2}\boldsymbol{B}_k\boldsymbol{\eta}_k\phi(\boldsymbol{x}_k)]\\ &\quad\times[\varepsilon_1^{1/2}(\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)\tilde{\boldsymbol{x}}_k - \varepsilon_1^{-1/2}\boldsymbol{B}_k\boldsymbol{\eta}_k\phi(\boldsymbol{x}_k)]^{\mathrm{T}} \ge 0\end{aligned} \tag{34}$$

yields

$$\begin{aligned}&(\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)\tilde{\boldsymbol{x}}_k(\boldsymbol{B}_k\boldsymbol{\eta}_k\phi(\boldsymbol{x}_k))^{\mathrm{T}}\\ &\quad+ \boldsymbol{B}_k\boldsymbol{\eta}_k\phi(\boldsymbol{x}_k)\tilde{\boldsymbol{x}}_k^{\mathrm{T}}(\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)^{\mathrm{T}}\\ &\le \varepsilon_1(\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)\tilde{\boldsymbol{x}}_k\tilde{\boldsymbol{x}}_k^{\mathrm{T}}(\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)^{\mathrm{T}}\\ &\quad+ \varepsilon_1^{-1}\boldsymbol{B}_k\boldsymbol{\eta}_k\phi(\boldsymbol{x}_k)\phi^{\mathrm{T}}(\boldsymbol{x}_k)\boldsymbol{\eta}_k^{\mathrm{T}}\boldsymbol{B}_k^{\mathrm{T}}\end{aligned} \tag{35}$$

Substituting (35) into (21) leads to

$$\begin{aligned}\boldsymbol{P}_{k+1|k} \le{}& (1+\varepsilon_1)(\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)\boldsymbol{P}_k(\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)^{\mathrm{T}}\\ &+ (1+\varepsilon_1^{-1})\boldsymbol{B}_k\mathrm{E}[\boldsymbol{\eta}_k\phi(\boldsymbol{x}_k)\phi^{\mathrm{T}}(\boldsymbol{x}_k)\boldsymbol{\eta}_k^{\mathrm{T}}]\boldsymbol{B}_k^{\mathrm{T}}\\ &+ \sum_{i=1}^{s}\boldsymbol{A}_{ik}\mathrm{E}(\boldsymbol{x}_k\boldsymbol{x}_k^{\mathrm{T}})\boldsymbol{A}_{ik}^{\mathrm{T}} + \boldsymbol{Q}_k\end{aligned} \tag{36}$$

Assume that ε_2 is a positive scalar. The matrix inequality can be obtained as

$$\tilde{\boldsymbol{x}}_k\hat{\boldsymbol{x}}_k^{\mathrm{T}} + \hat{\boldsymbol{x}}_k\tilde{\boldsymbol{x}}_k^{\mathrm{T}} \le \varepsilon_2\tilde{\boldsymbol{x}}_k\tilde{\boldsymbol{x}}_k^{\mathrm{T}} + \varepsilon_2^{-1}\hat{\boldsymbol{x}}_k\hat{\boldsymbol{x}}_k^{\mathrm{T}} \tag{37}$$

From (11) and (37), we have

$$\begin{aligned}\mathrm{E}(\boldsymbol{x}_k\boldsymbol{x}_k^{\mathrm{T}}) \le \mathrm{E}[(\tilde{\boldsymbol{x}}_k + \hat{\boldsymbol{x}}_k)(\tilde{\boldsymbol{x}}_k + \hat{\boldsymbol{x}}_k)^{\mathrm{T}}] \le (1+\varepsilon_2)\boldsymbol{P}_k\\ + (1+\varepsilon_2^{-1})\hat{\boldsymbol{x}}_k\hat{\boldsymbol{x}}_k^{\mathrm{T}}\end{aligned} \tag{38}$$

According to the Lemma 2, we obtain

$$\mathrm{E}[\boldsymbol{\eta}_k\phi(\boldsymbol{x}_k)\phi^{\mathrm{T}}(\boldsymbol{x}_k)\boldsymbol{\eta}_k^{\mathrm{T}}] \le \boldsymbol{\Xi}_k \tag{39}$$

Assume that there exists $\lambda > 0$ and the matrix $\boldsymbol{L}_k$ satisfying the inequality $0.5\lambda^{-1}\boldsymbol{I}_{n\times n} - \boldsymbol{L}_k\boldsymbol{P}_k\boldsymbol{L}_k^{\mathrm{T}} > 0$. According to the Lemma 1, we have

$$\begin{aligned}&(\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)\boldsymbol{P}_k(\boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k)^{\mathrm{T}}\\ &\quad\le \boldsymbol{F}_k(\boldsymbol{P}_k^{-1} - 2\lambda\boldsymbol{L}_k^{\mathrm{T}}\boldsymbol{L}_k)^{-1}\boldsymbol{F}_k^{\mathrm{T}} + \lambda^{-1}\boldsymbol{A}_k\boldsymbol{A}_k^{\mathrm{T}}\end{aligned} \tag{40}$$

Substituting (38)–(40) into (36), the upper bound of the one-step prediction covariance $\boldsymbol{P}_{k+1|k}$ can be written as

$$\begin{aligned}\boldsymbol{P}_{k+1|k} \le{}& (1+\varepsilon_1)[\boldsymbol{F}_k(\boldsymbol{P}_k^{-1} - 2\lambda\boldsymbol{L}_k^{\mathrm{T}}\boldsymbol{L}_k)^{-1}\boldsymbol{F}_k^{\mathrm{T}} + \lambda^{-1}\boldsymbol{A}_k\boldsymbol{A}_k^{\mathrm{T}}]\\ &+ (1+\varepsilon_1^{-1})\boldsymbol{B}_k\boldsymbol{\Xi}_k\boldsymbol{B}_k^{\mathrm{T}} + \boldsymbol{Q}_k\\ &+ \sum_{i=1}^{s}A_{ik}[(1+\varepsilon_2)\boldsymbol{P}_k + (1+\varepsilon_2^{-1})\hat{\boldsymbol{x}}_k\hat{\boldsymbol{x}}_k^{\mathrm{T}}]\boldsymbol{A}_{ik}^{\mathrm{T}}\end{aligned}$$

Similar to the derivation of (35), assume that ε_3 is a positive scalar, we obtain (see (41))

Substituting (41) into (30) yields

$$\begin{aligned}\boldsymbol{P}_{k+1} \le{}& (1+\varepsilon_3)(\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{\alpha}_{k+1}\boldsymbol{L}_{k+1})\\ &\times\boldsymbol{P}_{k+1|k}(\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{\alpha}_{k+1}\boldsymbol{L}_{k+1})^{\mathrm{T}}\\ &+ (1+\varepsilon_3^{-1})\boldsymbol{K}_{k+1}\boldsymbol{D}_{k+1}\mathrm{E}[\boldsymbol{\xi}_{k+1}g(\boldsymbol{x}_{k+1})g^{\mathrm{T}}(\boldsymbol{x}_{k+1})\boldsymbol{\xi}_{k+1}^{\mathrm{T}}]\\ &\times\boldsymbol{D}_{k+1}^{\mathrm{T}}\boldsymbol{K}_{k+1}^{\mathrm{T}} + \boldsymbol{K}_{k+1}\boldsymbol{R}_{k+1}\boldsymbol{K}_{k+1}^{\mathrm{T}}\\ &+ \boldsymbol{K}_{k+1}\sum_{i=1}^{r}\boldsymbol{C}_{ik+1}\mathrm{E}(\boldsymbol{x}_{k+1}\boldsymbol{x}_{k+1}^{\mathrm{T}})\boldsymbol{C}_{ik+1}^{\mathrm{T}}\boldsymbol{K}_{k+1}^{\mathrm{T}}\end{aligned} \tag{42}$$

Assume that ε_4 is a positive scalar, according to Lemma 2 and (38), we have

$$\mathrm{E}[\boldsymbol{\xi}_{k+1}g(\boldsymbol{x}_{k+1})g^{\mathrm{T}}(\boldsymbol{x}_{k+1})\boldsymbol{\xi}_{k+1}^{\mathrm{T}}] \le \boldsymbol{\Theta}_{k+1} \tag{43}$$

$$\mathrm{E}(\boldsymbol{x}_{k+1}\boldsymbol{x}_{k+1}^{\mathrm{T}}) \le (1+\varepsilon_4)\boldsymbol{P}_{k+1|k} + (1+\varepsilon_4^{-1})\hat{\boldsymbol{x}}_{k+1|k}\hat{\boldsymbol{x}}_{k+1|k}^{\mathrm{T}} \tag{44}$$

Assume that there exists $\mu > 0$ and the matrix $\boldsymbol{L}_{k+1}$ satisfying the inequality $\mu^{-1}\boldsymbol{I}_{n\times n} - \boldsymbol{L}_{k+1}\boldsymbol{P}_{k+1|k}\boldsymbol{L}_{k+1}^{\mathrm{T}} > 0$. According to the Lemma 1, we obtain (see (45))

$$\begin{aligned}&- (\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{\alpha}_{k+1}\boldsymbol{L}_{k+1})\tilde{\boldsymbol{x}}_{k+1|k}g^{\mathrm{T}}(\boldsymbol{x}_{k+1})\boldsymbol{\xi}_{k+1}^{\mathrm{T}}\boldsymbol{D}_{k+1}^{\mathrm{T}}\boldsymbol{K}_{k+1}^{\mathrm{T}}\\ &- \boldsymbol{K}_{k+1}\boldsymbol{D}_{k+1}\boldsymbol{\xi}_{k+1}g(\boldsymbol{x}_{k+1})\tilde{\boldsymbol{x}}_{k+1|k}^{\mathrm{T}}(\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{\alpha}_{k+1}\boldsymbol{L}_{k+1})^{\mathrm{T}}\\ &\le \varepsilon_3(\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{\alpha}_{k+1}\boldsymbol{L}_{k+1})\tilde{\boldsymbol{x}}_{k+1|k}\tilde{\boldsymbol{x}}_{k+1|k}^{\mathrm{T}}(\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{\alpha}_{k+1}\boldsymbol{L}_{k+1})^{\mathrm{T}}\\ &\quad+ \varepsilon_3^{-1}\boldsymbol{K}_{k+1}\boldsymbol{D}_{k+1}\boldsymbol{\xi}_{k+1}g(\boldsymbol{x}_{k+1})g^{\mathrm{T}}(\boldsymbol{x}_{k+1})\boldsymbol{\xi}_{k+1}^{\mathrm{T}}\boldsymbol{D}_{k+1}^{\mathrm{T}}\boldsymbol{K}_{k+1}^{\mathrm{T}}\end{aligned} \tag{41}$$

$$\begin{aligned}&(\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{\alpha}_{k+1}\boldsymbol{L}_{k+1})\boldsymbol{P}_{k+1|k}(\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{\alpha}_{k+1}\boldsymbol{L}_{k+1})^{\mathrm{T}}\\ &\quad\le (\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1})(\boldsymbol{P}_{k+1|k}^{-1} - \mu\boldsymbol{L}_{k+1}^{\mathrm{T}}\boldsymbol{L}_{k+1})^{-1}(\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1})^{\mathrm{T}} + \mu^{-1}\boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{C}_{k+1}^{\mathrm{T}}\boldsymbol{K}_{k+1}^{\mathrm{T}}\end{aligned} \tag{45}$$

doi: 10.1049/iet-cta.2014.0293

Substituting (43)–(45) into (42), the upper bound of the filtering error covariance P_{k+1} can be expressed as

$$\begin{aligned}\boldsymbol{P}_{k+1} \le{}& (1+\varepsilon_3)[(\boldsymbol{I}_{n\times n}-\boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1})(\boldsymbol{P}^{-1}_{k+1|k}-\mu\boldsymbol{L}^{\mathrm{T}}_{k+1}\boldsymbol{L}_{k+1})^{-1}\\ &\times(\boldsymbol{I}_{n\times n}-\boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1})^{\mathrm{T}}+\mu^{-1}\boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{C}^{\mathrm{T}}_{k+1}\boldsymbol{K}^{\mathrm{T}}_{k+1}]\\ &+(1+\varepsilon_3^{-1})\boldsymbol{K}_{k+1}\boldsymbol{D}_{k+1}\boldsymbol{\Theta}_{k+1}\boldsymbol{D}^{\mathrm{T}}_{k+1}\boldsymbol{K}^{\mathrm{T}}_{k+1}\\ &+\boldsymbol{K}_{k+1}\boldsymbol{R}_{k+1}\boldsymbol{K}^{\mathrm{T}}_{k+1}+\boldsymbol{K}_{k+1}\sum_{i=1}^{r}\boldsymbol{C}_{ik+1}[(1+\varepsilon_4)\boldsymbol{P}_{k+1|k}\\ &+(1+\varepsilon_4^{-1})\hat{\boldsymbol{x}}_{k+1|k}\hat{\boldsymbol{x}}^{\mathrm{T}}_{k+1|k}]\boldsymbol{C}^{\mathrm{T}}_{ik+1}\boldsymbol{K}^{\mathrm{T}}_{k+1}\end{aligned}$$

In order to use the recursive method to construct upper bounds of the filtering error covariance $\boldsymbol{P}_k(0\le k\le n)$, the Theorem 2 is given. □

Theorem 2: Consider the covariance matrices of the one-step prediction error and the filtering error in (32) and (33). Assume that the conditions shown in Theorem 1 come into existence. Let ε_1, ε_2, ε_3, ε_4, λ and μ be positive scalars. If the following two discrete-time Riccati difference equation

$$\begin{aligned}\boldsymbol{\Sigma}_{k+1|k} ={}& (1+\varepsilon_1)[\boldsymbol{F}_k(\boldsymbol{\Sigma}^{-1}_k-2\lambda\boldsymbol{L}^{\mathrm{T}}_k\boldsymbol{L}_k)^{-1}\boldsymbol{F}^{\mathrm{T}}_k+\lambda^{-1}\boldsymbol{A}_k\boldsymbol{A}^{\mathrm{T}}_k]\\ &+(1+\varepsilon_1^{-1})\boldsymbol{B}_k\boldsymbol{\Xi}_k\boldsymbol{B}^{\mathrm{T}}_k+\boldsymbol{Q}_k\\ &+\sum_{i=1}^{s}\boldsymbol{A}_{ik}[(1+\varepsilon_2)\boldsymbol{\Sigma}_k+(1+\varepsilon_2^{-1})\hat{\boldsymbol{x}}_k\hat{\boldsymbol{x}}^{\mathrm{T}}_k]\boldsymbol{A}^{\mathrm{T}}_{ik}\end{aligned}\quad(46)$$

$$\begin{aligned}\boldsymbol{\Sigma}_{k+1} ={}& (1+\varepsilon_3)[(\boldsymbol{I}_{n\times n}-\boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1})(\boldsymbol{\Sigma}^{-1}_{k+1|k}-\mu\boldsymbol{L}^{\mathrm{T}}_{k+1}\boldsymbol{L}_{k+1})^{-1}\\ &\times(\boldsymbol{I}_{n\times n}-\boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1})^{\mathrm{T}}+\mu^{-1}\boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{C}^{\mathrm{T}}_{k+1}\boldsymbol{K}^{\mathrm{T}}_{k+1}]\\ &+(1+\varepsilon_3^{-1})\boldsymbol{K}_{k+1}\boldsymbol{D}_{k+1}\boldsymbol{\Theta}_{k+1}\boldsymbol{D}^{\mathrm{T}}_{k+1}\boldsymbol{K}^{\mathrm{T}}_{k+1}\\ &+\boldsymbol{K}_{k+1}\boldsymbol{R}_{k+1}\boldsymbol{K}^{\mathrm{T}}_{k+1}+\boldsymbol{K}_{k+1}\sum_{i=1}^{r}\boldsymbol{C}_{ik+1}[(1+\varepsilon_4)\boldsymbol{\Sigma}_{k+1|k}\\ &+(1+\varepsilon_4^{-1})\hat{\boldsymbol{x}}_{k+1|k}\hat{\boldsymbol{x}}^{\mathrm{T}}_{k+1|k}]C^{\mathrm{T}}_{ik+1}\boldsymbol{K}^{\mathrm{T}}_{k+1}\end{aligned}\quad(47)$$

with initial covariance $\boldsymbol{\Sigma}_0 = P_0 > 0$ have positive-definite solution, the filtering error covariance $\boldsymbol{P}_{k+1}$ satisfies

$$\boldsymbol{P}_{k+1}\le\boldsymbol{\Sigma}_{k+1}(0\le k\le n)\quad(48)$$

Proof: Let $k=0$. Because of $\boldsymbol{\Sigma}_0=\boldsymbol{P}_0>0$, from (39) and (46), we have

$$\boldsymbol{P}_{1|0}\le\boldsymbol{\Sigma}_{1|0}\quad(49)$$

The inequality (49) yields

$$(\boldsymbol{P}^{-1}_{1|0}-\mu\boldsymbol{L}^{\mathrm{T}}_1\boldsymbol{L}_1)^{-1}\le(\boldsymbol{\Sigma}^{-1}_{1|0}-\mu\boldsymbol{L}^{\mathrm{T}}_1\boldsymbol{L}_1)^{-1}\quad(50)$$

Combining (45), (47) with (49), (50), it is easy to obtain that

$$\boldsymbol{P}_1\le\boldsymbol{\Sigma}_1\quad(51)$$

Let $k=n-1$. Assume that $\boldsymbol{P}_n\le\boldsymbol{\Sigma}_n$.
Let $k=n$. Owing to $\boldsymbol{P}_n\le\boldsymbol{\Sigma}_n$, there is

$$(\boldsymbol{P}^{-1}_k-2\lambda\boldsymbol{L}^{\mathrm{T}}_k\boldsymbol{L}_k)^{-1}\le(\boldsymbol{\Sigma}^{-1}_k-2\lambda\boldsymbol{L}^{\mathrm{T}}_k\boldsymbol{L}_k)^{-1}\quad(52)$$

From (39) and (46), there is $\boldsymbol{P}_{n+1|n}\le\boldsymbol{\Sigma}_{n+1|n}$. Hence, the result can be easily obtained

$$\boldsymbol{P}_{n+1}\le\boldsymbol{\Sigma}_{n+1}\quad(53)$$

According to the mathematical induction, (48) can be satisfied.

In order to obtain the filtering gain $\boldsymbol{K}_{k+1}$, the Theorem 3 is given. □

Theorem 3: To minimise the upper bounds, the filtering gain $\boldsymbol{K}_{k+1}$ is designed as

$$\begin{aligned}\boldsymbol{K}_{k+1} ={}& (1+\varepsilon_3)(\boldsymbol{\Sigma}^{-1}_{k+1|k}-\mu\boldsymbol{L}^{\mathrm{T}}_{k+1}\boldsymbol{L}_{k+1})^{-1}\boldsymbol{H}^{\mathrm{T}}_{k+1}\{(1+\varepsilon_3)\boldsymbol{H}_{k+1}\\ &\times(\boldsymbol{\Sigma}^{-1}_{k+1|k}-\mu\boldsymbol{L}^{\mathrm{T}}_{k+1}\boldsymbol{L}_{k+1})^{-1}\boldsymbol{H}^{\mathrm{T}}_{k+1}+\mu^{-1}\boldsymbol{C}_{k+1}\boldsymbol{C}^{\mathrm{T}}_{k+1}\\ &+(1+\varepsilon_3^{-1})\boldsymbol{D}_{k+1}\boldsymbol{\Theta}_{k+1}\boldsymbol{D}^{\mathrm{T}}_{k+1}+\boldsymbol{R}_{k+1}+\sum_{i=1}^{r}\boldsymbol{C}_{ik+1}\\ &\times[(1+\varepsilon_4)\boldsymbol{\Sigma}_{k+1|k}+(1+\varepsilon_4^{-1})\hat{\boldsymbol{x}}_{k+1|k}\hat{\boldsymbol{x}}^{\mathrm{T}}_{k+1|k}]\boldsymbol{C}^{\mathrm{T}}_{ik+1}\}^{-1}\end{aligned}\quad(54)$$

Proof: Constructing an optimised filtering gain $\boldsymbol{K}_{k+1}$ is to minimise the upper bound $\boldsymbol{\Sigma}_{k+1}$, according to (48), we have

$$\begin{aligned}\frac{\partial\boldsymbol{\Sigma}_{k+1}}{\partial\boldsymbol{K}_{k+1}} ={}& 2(1+\varepsilon_3)(\boldsymbol{I}_{n\times n}-\boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1})\\ &\times(\boldsymbol{\Sigma}^{-1}_{k+1|k}-\mu\boldsymbol{L}^{\mathrm{T}}_{k+1}\boldsymbol{L}_{k+1})^{-1}(-\boldsymbol{H}^{\mathrm{T}}_{k+1})\\ &+2\boldsymbol{K}_{k+1}\{\mu^{-1}\boldsymbol{C}_{k+1}\boldsymbol{C}^{\mathrm{T}}_{k+1}+\boldsymbol{R}_{k+1}\\ &+(1+\varepsilon_3^{-1})\boldsymbol{D}_{k+1}\boldsymbol{\Theta}_{k+1}\boldsymbol{D}^{\mathrm{T}}_{k+1}\\ &+\sum_{i=1}^{r}\boldsymbol{C}_{ik+1}[(1+\varepsilon_4)\boldsymbol{\Sigma}_{k+1|k}\\ &+(1+\varepsilon_4^{-1})\hat{\boldsymbol{x}}_{k+1|k}\hat{\boldsymbol{x}}^{\mathrm{T}}_{k+1|k}]\boldsymbol{C}^{\mathrm{T}}_{ik+1}\}\end{aligned}\quad(55)$$

Let $\partial\boldsymbol{\Sigma}_{k+1}/\partial\boldsymbol{K}_{k+1}=0$. The optimised filtering gain $\boldsymbol{K}_{k+1}$ can be obtained by (54). □

Remark 2: According to the above three theorems, the REKF can be accomplished by using (10), (46), (47) and (54). From the characterisation of the REKF, we can see that its structure resembles that of the EKF. In compared with the adaptive EKF (AEKF) in [2] and the robust Kalman filter (RKF) in [20], the main advantage of the proposed REKF is that in the case that there are multiplicative noises and unknown external disturbances for the attitude estimation system, the recursive REKF design is presented to compensate the two model uncertainties. The AEKF is not designed to control model uncertainties, which makes it become sensitive if there are multiplicative noises and unknown external disturbances in the system model. Nevertheless, the RKF gives an appropriate robust design to obtain a bounded solution of the Riccati difference equation by considering the multiplicative noises. In spite of this, unknown external disturbances frequently existing in the system are not taken into account in the RKF design. Thus, in order to compensate the measurement errors caused by unknown external disturbances, in presence of multiplicative noises, the REKF is derived to obtain the upper bound of the covariance matrices of the one-step prediction error and the filtering error and design the filtering gain $\boldsymbol{K}_{k+1}$ to minimise the upper bound. It is worth mentioning that the first and second terms in (46) and (47) arose from unknown external disturbances show the main difference between our work and the work in [20]. Meanwhile, the filter gain $\boldsymbol{K}_{k+1}$ in (55) can guarantee the minimised upper bound of the filtering error covariance. In this situation, the filter gain $\boldsymbol{K}_{k+1}=\boldsymbol{P}_{k+1|k}\boldsymbol{H}^{\mathrm{T}}_{k+1}(\boldsymbol{H}_{k+1}\boldsymbol{P}_{k+1|k}\boldsymbol{H}^{\mathrm{T}}_{k+1}+\boldsymbol{R}_{k+1})^{-1}$ of the AEKF and the counterpart in the RKF cannot make sure that the upper bound of the filtering error covariance in (47) is minimised.

In addition, the attitude estimation robust filtering problem with only multiplicative noises and unknown external disturbances is concerned. Some additional conditions can be assumed in the system model in order to accord with the realistic engineering applications, such as the measurement missing [27, 28] and the sensor delays [29], which will be one of our future research directions.

3.4 Implementation of the robust attitude estimation algorithm

For the non-linear attitude estimation system with model uncertainties in (3) and (8), the filter procedures of the robust attitude estimation algorithm are given as follows.

Step 1: Assume that the state estimate is $\hat{\boldsymbol{x}}_k$ at time k, and the upper bound of the filtering error covariance is Σ_k. The one-step state prediction $\hat{\boldsymbol{x}}_{k+1|k}$ and the upper bound of its covariance matrix are obtained by (10) and (46).
Step 2: From (55), the optimal filtering gain $\boldsymbol{K}_{k+1}$ can be given. Then utilising (10) and (47), the state estimate $\hat{\boldsymbol{x}}_{k+1}$ and the upper bound of the filtering error covariance $\boldsymbol{\Sigma}_{k+1}$ at time $k+1$ are obtained.
Step 3: For the state $\boldsymbol{x}_k = [q_k^{\mathrm{T}} \quad \beta_k^{\mathrm{T}}]^{\mathrm{T}}$, the quaternion part must obey a normalisation constraint $\|\boldsymbol{q}_k\|_2 = 1$ that has to be addressed in the robust attitude estimation algorithm. Hence, the state estimate $\hat{\boldsymbol{x}}_{k+1}$ will be normalised.
Step 4: Then, reuse the step 1–3, real time to estimate the system state.

4 Stability analysis

For the sake of verifying the stability and feasibility of the REKF, the stability analysis lemma is given.

Lemma 3 [30]*:* Assume that $\zeta_k (k^3 \geq 0)$ is a stochastic process. If there are $\upsilon_{\min} > 0$, $\upsilon_{\max} > 0$, $\rho > 0$ and $0 < \kappa \leq 1$, such that the stochastic process $V(\zeta_k)$ satisfies

$$\upsilon_{\min}\|\boldsymbol{\zeta}_k\|^2 \leq V(\boldsymbol{\zeta}_k) \leq \upsilon_{\max}\|\boldsymbol{\zeta}_k\|^2 \tag{56}$$

and

$$E[V(\boldsymbol{\zeta}_k)|\boldsymbol{\zeta}_{k-1}] - V(\boldsymbol{\zeta}_{k-1}) \leq \rho - \kappa V(\boldsymbol{\zeta}_{k-1}) \tag{57}$$

The stochastic process ζ_k is bounded in mean square for $k \geq 0$, that is, we have

$$E\{\|\boldsymbol{\zeta}_k\|^2\} \leq \frac{\upsilon_{\max}}{\upsilon_{\min}} E\{\|\boldsymbol{\zeta}_0\|^2\}(1-\kappa)^k + \frac{\rho}{\upsilon_{\min}} \sum_{i=1}^{k-1} (1-\kappa)^i \tag{58}$$

For the proposed REKF, in order to simplify the calculation of stability analysis, the following hypothesis is given as (see equation at the bottom of the page)

From (18) and (27), the state estimation error can be transformed as

$$\begin{aligned}\tilde{\boldsymbol{x}}_{k+1} &= (\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\bar{\boldsymbol{H}}_{k+1})\bar{\boldsymbol{F}}_k\tilde{\boldsymbol{x}}_k \\ &\quad + (\boldsymbol{I}_{n\times n} - \boldsymbol{K}_{k+1}\bar{\boldsymbol{H}}_{k+1})\bar{\boldsymbol{w}}_k - \boldsymbol{K}_{k+1}\bar{\boldsymbol{v}}_{k+1}\end{aligned} \tag{59}$$

The following theorem is given to illustrate the stability conditions of the REKF.

Theorem 4: Consider the uncertain non-linear system shown as (9) and the REKF. Assume that for $k \geq 0$, there are

$$\sigma \geq \sqrt{2}, \quad \boldsymbol{\Sigma}_k^{-1} \leq \boldsymbol{L}_k^{\mathrm{T}}\boldsymbol{L}_k \leq 2\boldsymbol{\Sigma}_k^{-1},$$
$$\det(\bar{\boldsymbol{F}}_k) \neq 0, \quad \det(\boldsymbol{I} - \boldsymbol{K}_k\bar{\boldsymbol{H}}_k) \neq 0$$

If there exist positive scalars $\varepsilon_{\min}$, $\varepsilon_{\max}$, $q_{\min}$, $q_{\max}$, $r_{\min}$, $r_{\max}$, $\phi_{\max}$, $\varphi_{\max}$, $b_{\max}$, $a_{\max}$ and $l_{\max}$, such that (see equation at the bottom of the page)

then there are $\rho > 0$ and $0 < \kappa \leq 1$, such that the estimation error is bounded in mean square, that is we have

$$E\{\|\tilde{\boldsymbol{x}}_k\|^2\} \leq \frac{\varepsilon_{\max}}{\varepsilon_{\min}} E\{\|\tilde{\boldsymbol{x}}_0\|^2\}(1-\kappa)^k + \frac{\rho}{\varepsilon_{\min}} \sum_{i=1}^{k-1} (1-\kappa)^i$$

Proof: A Lyapunov function is constructed as

$$V(\tilde{\boldsymbol{x}}_{k+1}) = \tilde{\boldsymbol{x}}_{k+1}^{\mathrm{T}}\boldsymbol{\Sigma}_{k+1}^{-1}\tilde{\boldsymbol{x}}_{k+1} \tag{60}$$

where, $\boldsymbol{\Sigma}_{k+1}$ is given by (47). According to the above conditions, we have

$$\frac{1}{\varepsilon_{\max}}\|\tilde{\boldsymbol{x}}_{k+1}\|^2 \leq V(\tilde{\boldsymbol{x}}_{k+1}) \leq \frac{1}{\varepsilon_{\min}}\|\tilde{\boldsymbol{x}}_{k+1}\|^2 \tag{61}$$

$$\bar{\boldsymbol{F}}_k = \boldsymbol{F}_k + \boldsymbol{A}_k\boldsymbol{\beta}_k\boldsymbol{L}_k, \quad \bar{\boldsymbol{H}}_{k+1} = \boldsymbol{H}_{k+1} + \boldsymbol{C}_{k+1}\boldsymbol{\alpha}_{k+1}\boldsymbol{L}_{k+1}$$

$$\bar{\boldsymbol{w}}_k = \boldsymbol{B}_k\boldsymbol{\eta}_k\phi(\boldsymbol{x}_k) + \sum_{i=1}^{s}\boldsymbol{A}_{ik}\boldsymbol{\eta}_{ik}\boldsymbol{x}_k + \boldsymbol{w}_k, \quad \bar{\boldsymbol{v}}_{k+1} = \boldsymbol{D}_{k+1}\boldsymbol{\xi}_{k+1}g(x_{k+1}) + \sum_{i=1}^{r}\boldsymbol{C}_{ik+1}\boldsymbol{\xi}_{ik+1}\boldsymbol{x}_{k+1} + \boldsymbol{v}_{k+1}$$

$$\bar{\boldsymbol{Q}}_k = (1+\varepsilon_1^{-1})\boldsymbol{B}_k\boldsymbol{\Xi}_k\boldsymbol{B}_k^{\mathrm{T}} + \boldsymbol{Q}_k + \sum_{i=1}^{s}\boldsymbol{A}_{ik}[(1+\varepsilon_2)\boldsymbol{\Sigma}_k + (1+\varepsilon_2^{-1})\hat{\boldsymbol{x}}_k\hat{\boldsymbol{x}}_k^{\mathrm{T}}]\boldsymbol{A}_{ik}^{\mathrm{T}}$$

$$\bar{\boldsymbol{R}}_{k+1} = (1+\varepsilon_3^{-1})\boldsymbol{D}_{k+1}\boldsymbol{\Theta}_{k+1}\boldsymbol{D}_{k+1}^{\mathrm{T}} + \boldsymbol{R}_{k+1} + \sum_{i=1}^{r}\boldsymbol{C}_{ik+1}[(1+\varepsilon_4)\boldsymbol{\Sigma}_{k+1|k} + (1+\varepsilon_4^{-1})\hat{\boldsymbol{x}}_{k+1|k}\hat{\boldsymbol{x}}_{k+1|k}^{\mathrm{T}}]\boldsymbol{C}_{ik+1}^{\mathrm{T}}$$

$$q_{\min}\boldsymbol{I}_{n\times n} \leq \boldsymbol{Q}_k \leq q_{\max}\boldsymbol{I}_{n\times n}, \quad r_{\min}\boldsymbol{I}_{m\times m} \leq \boldsymbol{R}_{k+1} \leq r_{\max}\boldsymbol{I}_{m\times m}, \quad \varepsilon_{\min}\boldsymbol{I}_{n\times n} \leq \Sigma_k \leq \varepsilon_{\max}\boldsymbol{I}_{n\times n}, \quad \hat{\boldsymbol{x}}_k\hat{\boldsymbol{x}}_k^{\mathrm{T}} \leq l_{\max}\boldsymbol{I}_{n\times n}$$
$$\boldsymbol{A}_k\boldsymbol{A}_k^{\mathrm{T}} \leq a_{\max}\boldsymbol{I}_{n\times n}, \quad \boldsymbol{B}_k\boldsymbol{B}_k^{\mathrm{T}} \leq b_{\max}\boldsymbol{I}_{n\times n}, \quad \boldsymbol{\Xi}_k \leq \phi_{\max}\boldsymbol{I}_{p\times p}, \quad \boldsymbol{\Theta}_{k+1} \leq \phi_{\max}\boldsymbol{I}_{q\times q}$$
$$\boldsymbol{A}_{ik}\boldsymbol{A}_{ik}^{\mathrm{T}} \leq \tau_{\max}\boldsymbol{I}_{n\times n}, \quad \boldsymbol{C}_{ik}\boldsymbol{C}_{ik}^{\mathrm{T}} \leq c_{\max}\boldsymbol{I}_{m\times m}, \quad \boldsymbol{F}_k\boldsymbol{F}_k^{\mathrm{T}} \leq f_{\max}\boldsymbol{I}_{n\times n}, \quad \boldsymbol{D}_k\boldsymbol{D}_k^{\mathrm{T}} \leq d_{\max}\boldsymbol{I}_{m\times m}$$

doi: 10.1049/iet-cta.2014.0293

If there are $\upsilon_{\min} = 1/\varepsilon_{\max}$ and $\upsilon_{\max} = 1/\varepsilon_{\min}$, the condition inequality (1) can be satisfied. According to Lemma 3, to prove the estimation error is bounded in mean square, it needs to verify that the upper bound on $E[V(\tilde{x}_{k+1})|\tilde{x}_k] - V(\tilde{x}_k)$ meets the (57). Hence, substituting (59) into (60), we have (see (62))

Similar to (35), the following matrix inequalities can be given as (see (63)–(65))

Inserting (8)–(10) into (7) yields

$$\begin{aligned} V(\tilde{x}_{k+1}) \le {} & \left(1+\frac{\sigma^{-2}}{2}\right)\tilde{x}_k^{\mathrm{T}}\bar{F}_k^{\mathrm{T}}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})^{\mathrm{T}}\Sigma_{k+1}^{-1} \\ & \times (I_{n\times n} - K_{k+1}\bar{H}_{k+1})\bar{F}_k\tilde{x}_k + (1+5\sigma^2)\bar{w}_k^{\mathrm{T}} \\ & \times (I_{n\times n} - K_{k+1}\bar{H}_{k+1})^{\mathrm{T}}\Sigma_{k+1}^{-1} \\ & \times (I_{n\times n} - K_{k+1}\bar{H}_{k+1})\bar{w}_k \\ & + (1+5\sigma^2)\bar{v}_{k+1}^{\mathrm{T}}K_{k+1}^{\mathrm{T}}\Sigma_{k+1}^{-1}K_{k+1}\bar{v}_{k+1} \end{aligned} \tag{66}$$

According to the Lemma 1, from (46) and (47), we obtain

$$\begin{aligned} \Sigma_{k+1} \ge {} & (1+\varepsilon_3)[(I_{n\times n} - K_{k+1}H_{k+1})(\Sigma_{k+1|k}^{-1} - \mu L_{k+1}^{\mathrm{T}}L_{k+1})^{-1} \\ & \times (I_{n\times n} - K_{k+1}H_{k+1})^{\mathrm{T}} + \mu^{-1}K_{k+1}C_{k+1}C_{k+1}^{\mathrm{T}}K_{k+1}^{\mathrm{T}}] \\ \ge {} & (1+\varepsilon_3)(I_{n\times n} - K_{k+1}\bar{H}_{k+1}) \\ & \times \Sigma_{k+1|k}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})^{\mathrm{T}} \end{aligned} \tag{67}$$

$$\begin{aligned} \Sigma_{k+1|k} \ge {} & (1+\varepsilon_1)[F_k(\Sigma_k^{-1} - 2\lambda L_k^{\mathrm{T}}L_k)^{-1}F_k^{\mathrm{T}} + \lambda^{-1}A_kA_k^{\mathrm{T}}] \\ \ge {} & (1+\varepsilon_1)[F_k\Sigma_k^{-1} - \lambda L_k^{\mathrm{T}}L_k - \lambda L_k^{\mathrm{T}}L_k)^{-1}F_k^{\mathrm{T}} \\ & + \lambda^{-1}A_kA_k^{\mathrm{T}}] \\ \ge {} & (1+\varepsilon_1)\bar{F}_k(\Sigma_k^{-1} - \lambda L_k^{\mathrm{T}}L_k)^{-1}\bar{F}_k^{\mathrm{T}} \end{aligned} \tag{68}$$

Assume that $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon$. Let $\lambda = \sigma^{-2}$. Then using the condition $L_k^{\mathrm{T}}L_k \ge \Sigma_k^{-1}$, from (67) and (68), we have (see (69)–(70))

To obtain the third term on the right-hand side of (66), according to (47) and (54), the filtering gain can be expressed as

$$K_{k+1} = \Sigma_{k+1}H_{k+1}^{\mathrm{T}}[(1+\varepsilon)\mu^{-1}C_{k+1}C_{k+1}^{\mathrm{T}} + \bar{R}_{k+1}]^{-1} \tag{71}$$

Combining (71) with (54) results in (see equation (72) at the bottom of next page)

Based on the above equation, we have

$$\begin{aligned} & \bar{v}_{k+1}^{\mathrm{T}}K_{k+1}^{\mathrm{T}}\Sigma_{k+1}^{-1}K_{k+1}\bar{v}_{k+1} \\ & = \bar{v}_{k+1}^{\mathrm{T}}[(1+\varepsilon)\mu^{-1}C_{k+1}C_{k+1}^{\mathrm{T}} + \bar{R}_{k+1}]^{-1}H_{k+1}\Sigma_{k+1}H_{k+1}^{\mathrm{T}} \\ & \quad \times [(1+\varepsilon)\mu^{-1}C_{k+1}C_{k+1}^{\mathrm{T}} + \bar{R}_{k+1}]^{-1}\bar{v}_{k+1} \\ & \le \bar{v}_{k+1}^{\mathrm{T}}[(1+\varepsilon)\mu^{-1}C_{k+1}C_{k+1}^{\mathrm{T}} + \bar{R}_{k+1}]^{-1}\bar{v}_{k+1} \\ & \le \bar{v}_{k+1}^{\mathrm{T}}\bar{R}_{k+1}^{-1}\bar{v}_{k+1} \end{aligned} \tag{73}$$

$$\begin{aligned} V(\tilde{x}_{k+1}) = {} & \tilde{x}_k^{\mathrm{T}}\bar{F}_k^{\mathrm{T}}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})^{\mathrm{T}}\Sigma_{k+1}^{-1}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})\bar{F}_k\tilde{x}_k + \bar{w}_k^{\mathrm{T}}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})^{\mathrm{T}}\Sigma_{k+1}^{-1}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})\bar{w}_k \\ & + \bar{v}_{k+1}^{\mathrm{T}}K_{k+1}^{\mathrm{T}}\Sigma_{k+1}^{-1}K_{k+1}\bar{v}_{k+1} + \tilde{x}_k^{\mathrm{T}}\bar{F}_k^{\mathrm{T}}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})^{\mathrm{T}}\Sigma_{k+1}^{-1}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})\bar{w}_k \\ & + \bar{w}_k^{\mathrm{T}}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})^{\mathrm{T}}\Sigma_{k+1}^{-1}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})\bar{F}_k\tilde{x}_k - \tilde{x}_k^{\mathrm{T}}\bar{F}_k^{\mathrm{T}}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})^{\mathrm{T}}\Sigma_{k+1}^{-1}K_{k+1}\bar{v}_{k+1} \\ & - \bar{v}_{k+1}^{\mathrm{T}}K_{k+1}^{\mathrm{T}}\Sigma_{k+1}^{-1}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})\bar{F}_k\tilde{x}_k - \bar{w}_k^{\mathrm{T}}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})^{\mathrm{T}}\Sigma_{k+1}^{-1}K_{k+1}\bar{v}_{k+1} \\ & - \bar{v}_{k+1}^{\mathrm{T}}K_{k+1}^{\mathrm{T}}\Sigma_{k+1}^{-1}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})\bar{w}_k \end{aligned} \tag{62}$$

$$\begin{aligned} & \tilde{x}_k^{\mathrm{T}}\bar{F}_k^{\mathrm{T}}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})^{\mathrm{T}}\Sigma_{k+1}^{-1}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})\bar{w}_k + \bar{w}_k^{\mathrm{T}}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})^{\mathrm{T}}\Sigma_{k+1}^{-1}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})\bar{F}_k\tilde{x}_k \\ & \quad \le \frac{\sigma^{-2}}{4}\tilde{x}_k^{\mathrm{T}}\bar{F}_k^{\mathrm{T}}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})^{\mathrm{T}}\Sigma_{k+1}^{-1}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})\bar{F}_k\tilde{x}_k + 4\sigma^2\bar{w}_k^{\mathrm{T}}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})^{\mathrm{T}}\Sigma_{k+1}^{-1}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})\bar{w}_k \end{aligned} \tag{63}$$

$$\begin{aligned} & -\tilde{x}_k^{\mathrm{T}}\bar{F}_k^{\mathrm{T}}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})^{\mathrm{T}}\Sigma_{k+1}^{-1}K_{k+1}\bar{v}_{k+1} - \bar{v}_{k+1}^{\mathrm{T}}K_{k+1}^{\mathrm{T}}\Sigma_{k+1}^{-1}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})\bar{F}_k\tilde{x}_k \\ & \quad \le \frac{\sigma^{-2}}{4}\tilde{x}_k^{\mathrm{T}}\bar{F}_k^{\mathrm{T}}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})^{\mathrm{T}}\Sigma_{k+1}^{-1}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})\bar{F}_k\tilde{x}_k + 4\sigma^2\bar{v}_{k+1}^{\mathrm{T}}K_{k+1}^{\mathrm{T}}\Sigma_{k+1}^{-1}K_{k+1}\bar{v}_{k+1} \end{aligned} \tag{64}$$

$$\begin{aligned} & -\bar{w}_k^{\mathrm{T}}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})^{\mathrm{T}}\Sigma_{k+1}^{-1}K_{k+1}\bar{v}_{k+1} - \bar{v}_{k+1}^{\mathrm{T}}K_{k+1}^{\mathrm{T}}\Sigma_{k+1}^{-1}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})\bar{w}_k \\ & \quad \le \sigma^{-2}\bar{w}_k^{\mathrm{T}}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})^{\mathrm{T}}\Sigma_{k+1}^{-1}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})\bar{w}_k + \sigma^2\bar{v}_{k+1}^{\mathrm{T}}K_{k+1}^{\mathrm{T}}\Sigma_{k+1}^{-1}K_{k+1}\bar{v}_{k+1} \end{aligned} \tag{65}$$

$$\begin{aligned} & \tilde{x}_k^{\mathrm{T}}\bar{F}_k^{\mathrm{T}}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})^{\mathrm{T}}\Sigma_{k+1}^{-1}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})\bar{F}_k\tilde{x}_k \\ & \quad \le (1+\varepsilon)^{-1}\tilde{x}_k^{\mathrm{T}}\bar{F}_k^{\mathrm{T}}\Sigma_{k+1|k}^{-1}\bar{F}_k\tilde{x}_k \le (1+\varepsilon)^{-2}\tilde{x}_k^{\mathrm{T}}(\Sigma_k^{-1} - \lambda L_k^{\mathrm{T}}L_k)\tilde{x}_k \le (1-\sigma^{-2})(1+\varepsilon)^{-2}\tilde{x}_k^{\mathrm{T}}\Sigma_k^{-1}\tilde{x}_k \end{aligned} \tag{69}$$

$$\begin{aligned} & \bar{w}_k^{\mathrm{T}}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})^{\mathrm{T}}\Sigma_{k+1}^{-1}(I_{n\times n} - K_{k+1}\bar{H}_{k+1})\bar{w}_k \\ & \quad \le (1+\varepsilon)^{-1}\bar{w}_k^{\mathrm{T}}\Sigma_{k+1|k}^{-1}\bar{w}_k = (1+\varepsilon)^{-1}\bar{w}_k^{\mathrm{T}}\{(1+\varepsilon)[F_k(\Sigma_k^{-1} - 2\lambda L_k^{\mathrm{T}}L_k)^{-1}F_k^{\mathrm{T}} + \lambda^{-1}A_kA_k^{\mathrm{T}}] + \bar{Q}_k\}^{-1}\bar{w}_k \\ & \quad \le (1+\varepsilon)^{-1}\bar{w}_k^{\mathrm{T}}\bar{Q}_k^{-1}\bar{w}_k \end{aligned} \tag{70}$$

Let ε be a small positive scalar. Substituting (69), (70) and (73) into (66) gives

$$\begin{aligned} V(\tilde{\boldsymbol{x}}_{k+1}) \le & \left(1+\frac{\sigma^{-2}}{2}\right)(1-\sigma^{-2})(1+\varepsilon)^{-2}\tilde{\boldsymbol{x}}_k^{\mathrm{T}}\boldsymbol{\Sigma}_k^{-1}\tilde{\boldsymbol{x}}_k \\ & + (1+5\sigma^2)(1+\varepsilon)^{-1}\bar{\boldsymbol{w}}_k^{\mathrm{T}}\bar{\boldsymbol{Q}}_k^{-1}\bar{\boldsymbol{w}}_k \\ & + (1+5\sigma^2)\bar{\boldsymbol{v}}_{k+1}^{\mathrm{T}}\bar{\boldsymbol{R}}_{k+1}^{-1}\bar{\boldsymbol{v}}_{k+1} \\ \le & \left(1-\frac{\sigma^{-2}}{2}-\frac{\sigma^{-4}}{2}\right)\tilde{\boldsymbol{x}}_k^{\mathrm{T}}\boldsymbol{\Sigma}_k^{-1}\tilde{\boldsymbol{x}}_k \\ & + (1+5\sigma^2)\bar{\boldsymbol{w}}_k^{\mathrm{T}}\bar{\boldsymbol{Q}}_k^{-1}\bar{\boldsymbol{w}}_k \\ & + (1+5\sigma^2)\bar{\boldsymbol{v}}_{k+1}^{\mathrm{T}}\bar{\boldsymbol{R}}_{k+1}^{-1}\bar{\boldsymbol{v}}_{k+1} \end{aligned} \tag{74}$$

Hence

$$\begin{aligned} E[V(\tilde{\boldsymbol{x}}_{k+1})|\tilde{\boldsymbol{x}}_k] - V(\tilde{\boldsymbol{x}}_k) \le & \left(-\frac{\sigma^{-2}}{2}-\frac{\sigma^{-4}}{2}\right)V(\tilde{\boldsymbol{x}}_k) \\ & + (1+5\sigma^2)E(\bar{\boldsymbol{w}}_k^{\mathrm{T}}\bar{\boldsymbol{Q}}_k^{-1}\bar{\boldsymbol{w}}_k) \\ & + (1+5\sigma^2)E(\bar{\boldsymbol{v}}_{k+1}^{\mathrm{T}}\bar{\boldsymbol{R}}_{k+1}^{-1}\bar{\boldsymbol{v}}_{k+1}) \end{aligned} \tag{75}$$

According to the definition of $\bar{\boldsymbol{w}}_k$ and $\bar{\boldsymbol{Q}}_k$, and the condition assumptions in the Theorem 4, we have

$$\begin{aligned} \bar{\boldsymbol{Q}}_k = & (1+\varepsilon^{-1})\boldsymbol{B}_k\boldsymbol{\Xi}_k\boldsymbol{B}_k^{\mathrm{T}} + \boldsymbol{Q}_k \\ & + \sum_{i=1}^{s}\boldsymbol{A}_{ik}[(1+\varepsilon)\boldsymbol{\Sigma}_k + (1+\varepsilon^{-1})\hat{\boldsymbol{x}}_k\hat{\boldsymbol{x}}_k^{\mathrm{T}}]\boldsymbol{A}_{ik}^{\mathrm{T}} \\ \le & \{(1+\varepsilon^{-1})(b_{\max}\phi_{\max} + s\tau_{\max}l_{\max}) \\ & + [1+(1+\varepsilon)s\tau_{\max}]q_{\max}\}\boldsymbol{I}_{n\times n} \end{aligned} \tag{76}$$

From (76), it is easy to have

$$\begin{aligned} \mathrm{E}(\bar{\boldsymbol{w}}_k^{\mathrm{T}}\bar{\boldsymbol{Q}}_k^{-1}\bar{\boldsymbol{w}}_k) \le & \mathrm{E}(\bar{\boldsymbol{w}}_k^{\mathrm{T}}\boldsymbol{Q}_k^{-1}\bar{\boldsymbol{w}}_k) \le \frac{1}{q_{\min}}\mathrm{E}(\bar{\boldsymbol{w}}_k^{\mathrm{T}}\bar{\boldsymbol{w}}_k) \\ = & \frac{1}{q_{\min}}\mathrm{tr}[\mathrm{E}(\bar{\boldsymbol{w}}_k\bar{\boldsymbol{w}}_k^{\mathrm{T}})] \le \frac{1}{q_{\min}}\mathrm{tr}(\bar{\boldsymbol{Q}}_k) \\ \le & \frac{n}{q_{\min}}\{(1+\varepsilon^{-1})(b_{\max}\phi_{\max} + s\tau_{\max}l_{\max}) \\ & + [1+(1+\varepsilon)s\tau_{\max}]q_{\max}\} \end{aligned} \tag{77}$$

Then, according to (10) and the assumption $\hat{\boldsymbol{x}}_k\hat{\boldsymbol{x}}_k^{\mathrm{T}} \le l_{\max}\boldsymbol{I}_{n\times n}$, the inequality $\hat{\boldsymbol{x}}_{k+1|k}\hat{\boldsymbol{x}}_{k+1|k}^{\mathrm{T}} \le m_{\max}\boldsymbol{I}_{n\times n}$ can be satisfied ($m_{\max}$ is a positive scalar).

Considering (46) and the condition $\boldsymbol{L}_k^{\mathrm{T}}\boldsymbol{L}_k \le 2\boldsymbol{\Sigma}_k^{-1}$, we have

$$\begin{aligned} \boldsymbol{\Sigma}_{k+1|k} \le & (1+\varepsilon)[(1-2\sigma^{-2})\boldsymbol{F}_k\boldsymbol{\Sigma}_k\boldsymbol{F}_k^{\mathrm{T}} + \sigma^2\boldsymbol{A}_k\boldsymbol{A}_k^{\mathrm{T}}] \\ & + (1+\varepsilon^{-1})\boldsymbol{B}_k\boldsymbol{\Xi}_k\boldsymbol{B}_k^{\mathrm{T}} + \boldsymbol{Q}_k \\ & + \sum_{i=1}^{s}\boldsymbol{A}_{ik}[(1+\varepsilon)\boldsymbol{\Sigma}_k + (1+\varepsilon^{-1})\hat{\boldsymbol{x}}_k\hat{\boldsymbol{x}}_k^{\mathrm{T}}]\boldsymbol{A}_{ik}^{\mathrm{T}} \\ \le & \{(1+\varepsilon)[(1-2\sigma^{-2})f_{\max}q_{\max} + \sigma^2 a_{\max}] + q_{\max} \\ & + (1+\varepsilon^{-1})b_{\max}\phi_{\max} + s\tau_{\max}[(1+\varepsilon)q_{\max} \\ & + (1+\varepsilon^{-1})l_{\max}]\}\boldsymbol{I}_{n\times n} \\ = & \theta\boldsymbol{I}_{n\times n} \end{aligned} \tag{78}$$

where $\theta = (1+\varepsilon)[(1-2\sigma^{-2})f_{\max}q_{\max} + \sigma^2 a_{\max}] + q_{\max} + s\tau_{\max}[(1+\varepsilon)q_{\max} + (1+\varepsilon^{-1})l_{\max}] > 0$.

According to (78), the condition assumptions and the definition of the $\bar{v}_{k+1}$ and $\bar{\boldsymbol{R}}_{k+1}$, we have

$$\begin{aligned} \bar{\boldsymbol{R}}_{k+1} = & (1+\varepsilon^{-1})\boldsymbol{D}_{k+1}\boldsymbol{\Theta}_{k+1}\boldsymbol{D}_{k+1}^{\mathrm{T}} + \boldsymbol{R}_{k+1} \\ & + \sum_{i=1}^{r}\boldsymbol{C}_{ik+1}[(1+\varepsilon)\boldsymbol{\Sigma}_{k+1|k} \\ & + (1+\varepsilon^{-1})\hat{\boldsymbol{x}}_{k+1|k}\hat{\boldsymbol{x}}_{k+1|k}^{\mathrm{T}}]C_{ik+1}^{\mathrm{T}} \\ \le & \{(1+\varepsilon^{-1})d_{\max}\varphi_{\max} + r_{\max} + rc_{\max}[(1+\varepsilon)\theta \\ & + (1+\varepsilon^{-1})m_{\max}]\}\boldsymbol{I}_{m\times m} \end{aligned} \tag{79}$$

Hence, we have

$$\begin{aligned} \mathrm{E}(\bar{\boldsymbol{v}}_{k+1}^{\mathrm{T}}\bar{\boldsymbol{R}}_{k+1}^{-1}\bar{\boldsymbol{v}}_{k+1}) \le & \mathrm{E}(\bar{\boldsymbol{v}}_{k+1}^{\mathrm{T}}\boldsymbol{R}_{k+1}^{-1}\bar{\boldsymbol{v}}_{k+1}) \le \frac{1}{r_{\min}}\mathrm{E}(\bar{\boldsymbol{v}}_{k+1}^{\mathrm{T}}\bar{\boldsymbol{v}}_{k+1}) \\ = & \frac{1}{r_{\min}}\mathrm{tr}[E(\bar{\boldsymbol{v}}_{k+1}\bar{\boldsymbol{v}}_{k+1}^{\mathrm{T}})] \le \frac{1}{r_{\min}}\mathrm{tr}(\bar{\boldsymbol{R}}_{k+1}) \\ \le & \frac{m}{r_{\min}}[(1+\varepsilon^{-1})(d_{\max}\varphi_{\max} + rc_{\max}m_{\max}) \\ & + r_{\max} + (1+\varepsilon)rc_{\max}\theta] \end{aligned} \tag{80}$$

Inserting (77) and (80) into (75) obtains

$$E[V(\tilde{\boldsymbol{x}}_{k+1})|\tilde{\boldsymbol{x}}_k] - V(\tilde{\boldsymbol{x}}_k) \le \rho - \kappa V(\tilde{\boldsymbol{x}}_k) \tag{81}$$

where $\kappa = (\sigma^{-2}/2) + (\sigma^{-4}/2)$

$$\begin{aligned} \rho = & \frac{n(1+5\sigma^2)}{q_{\min}}\{(1+\varepsilon^{-1})(b_{\max}\phi_{\max} + s\tau_{\max}l_{\max}) \\ & + [1+(1+\varepsilon)s\tau_{\max}]q_{\max}\} + \frac{m(1+5\sigma^2)}{r_{\min}} \\ & \times [(1+\varepsilon^{-1})(d_{\max}\varphi_{\max} + rc_{\max}m_{\max}) + r_{\max} \\ & + (1+\varepsilon)rc_{\max}\theta] \end{aligned}$$

$$\begin{aligned} & [(1+\varepsilon)\mu^{-1}\boldsymbol{C}_{k+1}\boldsymbol{C}_{k+1}^{\mathrm{T}} + \bar{\boldsymbol{R}}_{k+1}]^{-1} - [(1+\varepsilon)\mu^{-1}\boldsymbol{C}_{k+1}\boldsymbol{C}_{k+1}^{\mathrm{T}} + \bar{\boldsymbol{R}}_{k+1}]^{-1}\boldsymbol{H}_{k+1}\boldsymbol{\Sigma}_{k+1}\boldsymbol{H}_{k+1}^{\mathrm{T}}[(1+\varepsilon)\mu^{-1}\boldsymbol{C}_{k+1}\boldsymbol{C}_{k+1}^{\mathrm{T}} + \bar{\boldsymbol{R}}_{k+1}]^{-1} \\ & = [(1+\varepsilon)\mu^{-1}\boldsymbol{C}_{k+1}\boldsymbol{C}_{k+1}^{\mathrm{T}} + \bar{\boldsymbol{R}}_{k+1}]^{-1}(\boldsymbol{I}_{m\times m} - \boldsymbol{H}_{k+1}\boldsymbol{K}_{k+1}) \\ & = [(1+\varepsilon)\mu^{-1}\boldsymbol{C}_{k+1}\boldsymbol{C}_{k+1}^{\mathrm{T}} + \bar{\boldsymbol{R}}_{k+1}]^{-1}\{\boldsymbol{I}_{m\times m} - (1+\varepsilon)\boldsymbol{H}_{k+1}(\boldsymbol{\Sigma}_{k+1|k}^{-1} - \mu\boldsymbol{L}_{k+1}^{\mathrm{T}}\boldsymbol{L}_{k+1})^{-1}\boldsymbol{H}_{k+1}^{\mathrm{T}}[(1+\varepsilon)\boldsymbol{H}_{k+1} \\ & \quad \times (\boldsymbol{\Sigma}_{k+1|k}^{-1} - \mu\boldsymbol{L}_{k+1}^{\mathrm{T}}\boldsymbol{L}_{k+1})^{-1}\boldsymbol{H}_{k+1}^{\mathrm{T}} + (1+\varepsilon)\mu^{-1}\boldsymbol{C}_{k+1}\boldsymbol{C}_{k+1}^{\mathrm{T}} + \bar{\boldsymbol{R}}_{k+1}]^{-1}\} \\ & = [(1+\varepsilon)\boldsymbol{H}_{k+1}(\boldsymbol{\Sigma}_{k+1|k}^{-1} - \mu\boldsymbol{L}_{k+1}^{\mathrm{T}}\boldsymbol{L}_{k+1})^{-1}\boldsymbol{H}_{k+1}^{\mathrm{T}} + (1+\varepsilon)\mu^{-1}\boldsymbol{C}_{k+1}\boldsymbol{C}_{k+1}^{\mathrm{T}} + \bar{\boldsymbol{R}}_{k+1}]^{-1} > 0 \end{aligned} \tag{72}$$

doi: 10.1049/iet-cta.2014.0293

According to the assumption conditions, we can see that

$$\rho > 0, \quad 0 < \kappa < 1$$

Thus, the condition inequality (57) can be satisfied. Using the Lemma 3, the estimation error is bounded in mean square, that is, we have

$$E\{\|\tilde{x}_k\|^2\} \le \frac{\varepsilon_{\max}}{\varepsilon_{\min}} E\{\|\tilde{x}_0\|^2\}(1-\kappa)^k + \frac{\rho}{\varepsilon_{\min}} \sum_{i=1}^{k-1} (1-\kappa)^i$$

□

Remark 3: The stability analysis of the REKF for attitude estimation system with multiplicative noises and unknown external disturbances is given in the Theorem 4. The theoretical proof illustrates that the estimation error of the REKF is bounded in mean square when the conditions in the Theorem 4 hold. From the REKF algorithm, we can see that the matrices $\boldsymbol{A}_k$, $\boldsymbol{C}_{k+1}$ and $\boldsymbol{L}_k$, and the parameters ε, λ and μ are unknown. Therefore there is need to analysis the chosen bound of these matrices and parameters. According to (17) and (25), the unknown matrices β_k and α_{k+1} used to express the linearisation errors of the system model are bounded, but the magnitude of the known scaling matrices $\boldsymbol{A}_k$ and $\boldsymbol{C}_{k+1}$ have no limitations. Hence, even if there are large linearisation errors in the system, the conditions in (17) and (25) can be satisfied. The condition $\boldsymbol{\Sigma}_k^{-1} \le \boldsymbol{L}_k^{\mathrm{T}}\boldsymbol{L}_k \le 2\boldsymbol{\Sigma}_k^{-1}$ shows the bound of the tuning matrix $\boldsymbol{L}_k$. The parameter ε is set as a small positive scalar in the proof, and its related discussion can be found in [25]. Then, utilising the conditions $\sigma \ge \sqrt{2}$, $\lambda > 0$ and $\lambda = \sigma^{-2}$, the condition $0 < \lambda \le 0.5$ can be obtained. As the parameter μ is similar to the parameter λ, we can assume $\mu = \lambda$.

5 Simulated experiments and analysis

To verify the validity of the algorithm, some simulation cases on different initial conditions and simulation parameters are given to compare the proposed REKF algorithm with the AEKF algorithm in literature [2] and the RKF algorithm in literature [11]. For a fair comparison, the root-mean square error (RMSE) of the attitude is chosen to show the quality of the attitude estimation. In all simulations, a total of 50 Monte-Carlo simulation runs are set (i.e. $N_{MC} = 50$). Then, the RMSE in attitude angle is defined by [19]

$$\mathrm{RMSE}_{\mathrm{att}}(k) = \sqrt{\frac{1}{N_{\mathrm{MC}}} \sum_{i=1}^{N_{\mathrm{MC}}} \|\boldsymbol{ae}_i(k)\|^2} \tag{82}$$

where $\boldsymbol{ae}_i(k)$ is the attitude error vector at the ith Monte-Carlo run.

For completing the design of the REKF, the matrices $\boldsymbol{A}_k$, $\boldsymbol{C}_{k+1}$ and $\boldsymbol{L}_k$, and the parameters ε, λ and μ should be determined properly. Because of the high precision of the star sensors, the estimation error is rather small in the attitude estimation system. Therefore the linearisation error can be negligible and set $\boldsymbol{A}_k = \boldsymbol{0}_{n\times n}$, $\boldsymbol{C}_{k+1} = \boldsymbol{0}_{m\times n}$. The tuning matrix $\boldsymbol{L}_k$ is selected as $\boldsymbol{L}_k = \sqrt{\boldsymbol{\Sigma}_k^{-1}}$, which satisfies the condition in the Theorem 4. To ensure the estimation precision, the parameters λ and μ are set as 0.0001, and the parameter ε is set as 0.1. Since the measurement model contains additive noises only and the state model does not contain unknown disturbances, let $\boldsymbol{C}_{ik+1} = \boldsymbol{0}_{m\times n}$ and $\boldsymbol{B}_k = \boldsymbol{0}_{n\times p}$. The measurement model parameters $\boldsymbol{\sigma}_{ij}(i = 1, 2, 3; j = x, y, z)$ are set as 10 s. Table 1 contains all the setting parameters used in our experiments.

5.1 Case 1

Suppose a spacecraft is spinning at the constant angular velocity of $w(t) = \sin(2\pi t/150) \times [1 \quad -1 \quad 1]^{\mathrm{T}}$ deg/s. The initial attitude quaternion of the system is taken as $\boldsymbol{q}_0 = [0 \quad 0 \quad 0 \quad 1]^{\mathrm{T}}$. The sampling intervals of gyro are assumed as $\Delta t = 0.25$ s, and the simulation time N is set as 1000 s. According to (1), a gyro simulator is used to generate the gyro angular rate measurements. The standard deviation of gyros' measurement noise is $\sigma_v = 2.6875 \times 10^{-7}$ rad/s$^{1/2}$. The standard deviation of gyros' drift noise is $\sigma_u = 8.9289 \times 10^{-10}$ rad/s$^{3/2}$. The gyros' initial bias is set as $\beta = [0.1 \quad 0.1 \quad 0.1]^{\mathrm{T}}$ deg/h. Owing to using three star sensors, the measurements of these sensors are generated by using (8). The sampling frequency of star sensors is 1 Hz. The standard deviation of star sensors' measurement noise is all $\sigma_s = 18$ s. The reference vectors of the star sensors are set as $\vec{r}^1 = [1 \quad 0 \quad 0]^{\mathrm{T}}$, $\vec{r}^2 = [0 \quad 1 \quad 0]^{\mathrm{T}}$ and $\vec{r}^3 = [0 \quad 0 \quad 1]^{\mathrm{T}}$. The unknown disturbance vectors in the measurement are set to [−5 s, 5 s]. For this case, all the filters are initialised with no attitude errors and zero bias estimate errors. The initial

Table 1 Simulation Parameters in different cases

	Case 1	Case 2	Case 3
angular velocity $w(t)$, deg/s	$\sin(2\pi t/150) \times [1 \quad -1 \quad 1]^{\mathrm{T}}$	$\sin(2\pi t/150) \times [1 \quad -1 \quad 1]^{\mathrm{T}}$	$\sin(2\pi t/150) \times [1 \quad -1 \quad 1]^{\mathrm{T}}$
Gyro noise σ_v, rad/s$^{1/2}$	2.6875×10^{-7}	2.6875×10^{-7}	2.6875×10^{-7}
drift noise σ_u, rad/s$^{3/2}$	8.9289×10^{-10}	8.9289×10^{-10}	8.9289×10^{-10}
star sensor noise σ_s, s	18	18	18
initial attitude error, deg	$[0 \quad 0 \quad 0]^{\mathrm{T}}$	$[-10 \quad 10 \quad 20]^{\mathrm{T}}$	$[0 \quad 0 \quad 0]^{\mathrm{T}}$
initial bias estimate, deg/h	$[0.1 \quad 0.1 \quad 0.1]^{\mathrm{T}}$	$[0 \quad 0 \quad 0]^{\mathrm{T}}$	$[0.1 \quad 0.1 \quad 0.1]^{\mathrm{T}}$
initial attitude covariance, (deg)^2	0.01	1	0.01
initial bias covariance, (deg/h)^2	0.04	0.04	0.04
λ, μ	0.0001	0.0001	0.0001
ε	0.1	0.1	0.1
σ_{ij}, s	10	10	10
unknown disturbance (sec)	random numbers in [−5, 5]	random numbers in [−5, 5]	see the formula (83)
simulation time (s)	1000	2400	1000

attitude error covariance is set at $(0.1\text{deg})^2$ for the quaternion components and $(0.2\,\text{deg}/h)^2$ for the bias components.

A comparison of the quaternion estimation errors and the square root of the corresponding diagonal elements of the error covariance in AEKF, RKF and REKF is shown in Fig. 1. The quaternion estimation errors are expressed by red lines, and blue dashed lines show the square root of the corresponding diagonal elements of the error covariance. From the Fig. 1, it can be seen that the estimation errors of the quaternion vector part are beyond the bound of the computed covariance in AEKF and RKF. Meanwhile, it is obvious to be seen that the estimation errors in REKF are in the bound of the calculated covariance. This is because the AEKF is not suitable for handling model uncertainties of the multiplicative noises and unknown external disturbances. The effect of multiplicative noises is mitigated by using the RKF that introduces the robust design, but the unknown external disturbances in the measurement are not taken into account. The model uncertainties may degrade the filtering performance and cause biased estimate, even the divergence of the filter. Compared with the AEKF and RKF, the REKF can effectively deal with the attitude estimation problem with multiplicative noises and unknown external disturbances. This indicates that the estimation errors in REKF are bounded in mean square to ensure the stability of the REKF.

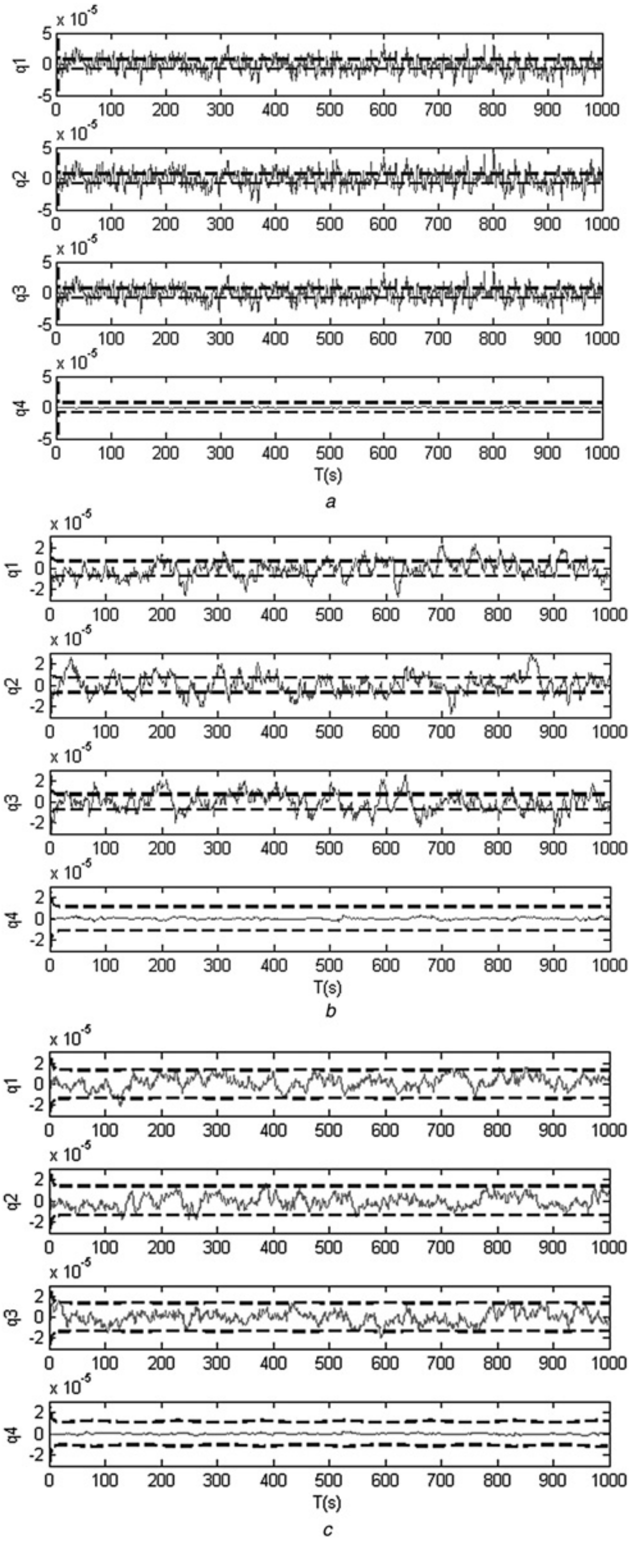

Fig. 1 *Comparison of the quaternion estimation errors (red solid line) and the square root of the corresponding diagonal elements of the error covariance (blue dashed line) in different filtering algorithms*

a Quaternion estimation errors in AEKF
b Quaternion estimation errors in RKF
c Quaternion estimation errors in REKF

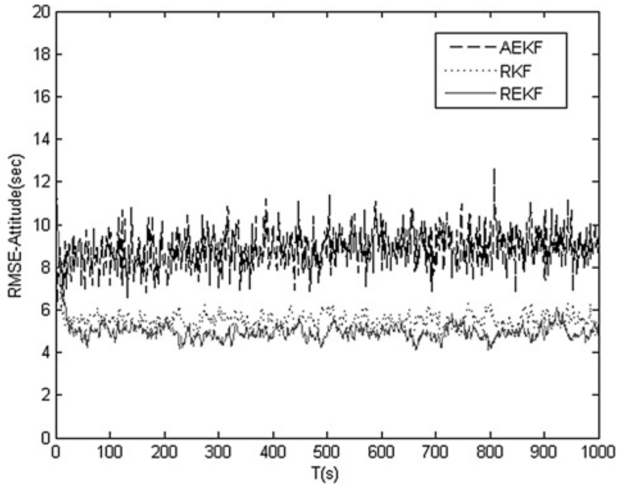

Fig. 2 *RMSE of the attitude angles in the case 1*

Furthermore, it is very important for attitude estimation to obtain the angle information, so the estimated quaternion needs to be converted as the form of Euler angles. To evaluate the estimation precision of the algorithm well, according to (82), the RMSE is employed to describe the quality of the Euler Angle estimation. A plot of the RMSE of the attitude angles is shown in Fig. 2. Obviously, it can be seen that the RKF has higher estimation precision than the AEKF, whereas the REKF is superior to the AEKF and RKF. This illustrates that the AEKF is sensitive to multiplicative noises and unknown external disturbances and the RKF is only sensitive to unknown external disturbances, whereas the REKF designed with multiplicative noises and unknown external disturbances is efficient to control the unfavourable effect of the two model uncertainties.

5.2 Case 2

In this case, a considerably condition is demonstrated using the different initial estimation errors, and the other same simulation parameters are set as in the case 1. The initial estimation errors are set to $-10°$, $10°$ and $20°$ for each axis, respectively, and the initial bias estimates are set to zero. The

doi: 10.1049/iet-cta.2014.0293

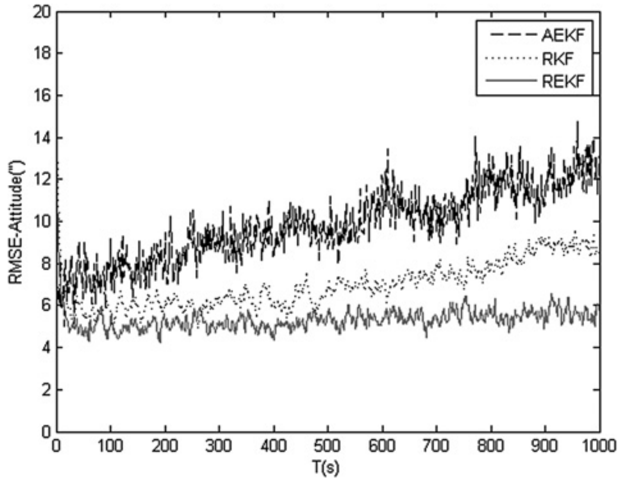

Fig. 3 *RMSE of the attitude angles in the case 2*

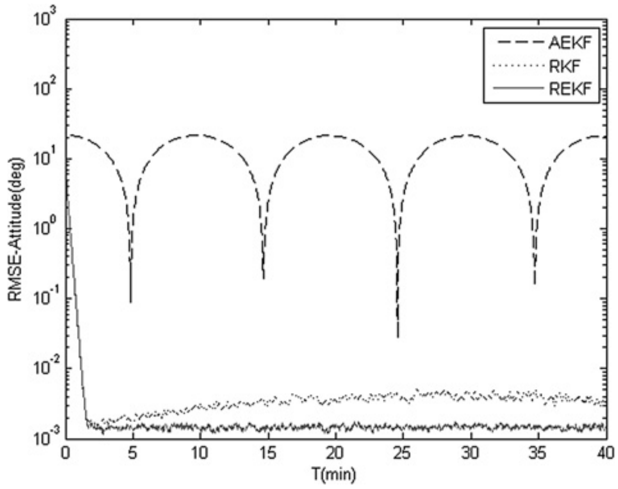

Fig. 4 *RMSE of the attitude angles in the case 3*

initial attitude covariance is set to $(1°)^2$ and the initial bias covariance is unchanged. The RMSE of the attitude angles for this simulation case is shown in Fig. 3. As can be seen, the convergence performance of the RKF and the REKF is remarkably superior to that of the AEKF. In the case that there are large initial attitude and bias estimation errors, the AEKF cannot achieve a converged solution, whereas the REKF performs the best in this case. Not surprisingly, model uncertainties and non-linearities affect the convergence and precision of the AEKF and RKF. For the REKF, this is because of the fact that the specific efforts are made to compensate the effects of non-linearities, multiplicative noises and unknown external disturbances.

5.3 Case 3

In the case 1 and 2, the unknown external disturbances are assumed as random numbers in bounded. However, because of the spacecraft body librating, the measurement errors caused by the unknown external disturbances may be cumulative. Therefore in this case, the unknown external disturbance errors can be expressed as

$$\dot{\varphi}_{ix} = p_1, p_1 = \begin{bmatrix} 1.5 \times 10^{-8} \\ 1.5 \times 10^{-8} \\ 10^{-8} \end{bmatrix}$$

$$\dot{\varphi}_{iy} = p_2, p_2 = \begin{bmatrix} 10^{-8} \\ 1.6 \times 10^{-8} \\ 1.6 \times 10^{-8} \end{bmatrix}$$

$$\dot{\varphi}_{iz} = p_3, p_3 = \begin{bmatrix} 1.9 \times 10^{-8} \\ 10^{-8} \\ 1.9 \times 10^{-8} \end{bmatrix} \tag{83}$$

Then, the simulation condition is the same as that of the case 1. The attitude estimation results of the case 3 are shown in Fig. 4. As shown in Fig. 4, compared with the case 1, the RMSE of the AEKF and RKF does not converge. The reason is that because of the influence of the cumulative unknown external disturbances, the AEKF and RKF cannot effectively capture these cumulative estimation errors. Different from the AEKF and RKF, the design of the REKF adequately considers the unknown external disturbance errors, so that it is appropriate to the situation in the case 3. Thus, the precision and the convergence performance of the REKF are much better.

6 Conclusion

In the fact that the existing attitude estimation algorithm is difficult to deal with the attitude estimation problem with multiplicative noises and unknown external disturbances, a REKF algorithm is developed in this paper for this system. The structure of the REKF is constructed, the algorithm is designed to minimise the upper bound on the state estimation error variance, and the algorithm stability is analysed. Finally, the effectiveness and applicability of the addressed algorithm has been demonstrated by the simulation.

7 References

1 Lefferts, E.J., Markley, F.L., Shuster, M.D.: 'Kalman filtering for spacecraft attitude estimation', *J. Guid. Control Dyn.*, 1982, **5**, (5), pp. 417–429
2 Bar-Itzhack, I.Y., Oshman, Y.: 'Attitude determination from vector observation: quaternion estimation', *IEEE Trans. Aerosp. Electron. Syst.*, 1985, **AES-21**, (1), pp. 128–136
3 Choukroun, D., Bar-Itzhack, I.Y., Oshman, Y.: 'Novel quaternion Kalman filter', *IEEE Trans. Aerosp. Electron. Syst.*, 2006, **42**, (1), pp. 174–190
4 Psiaki, M.L.: 'Backward-smoothing extended Kalman filter', *J. Guid. Control Dyn.*, 2005, **28**, (5), pp. 885–894
5 Zanetti, R., Majji, M., Bishop, R.H., Mortari, D.: 'Norm-constrained Kalman filtering', *J. Guid. Control Dyn.*, 2009, **32**, (5), pp. 1458–1465
6 Li, P., Lam, J., Shu, Z.: 'H_∞ positive filtering for positive linear discrete-time systems: an augmentation approach', *IEEE Trans. Autom. Control*, 2010, **55**, (10), pp. 2337–2342
7 Wu, L., Zheng, W.: 'Weighted H_∞ model reduction for linear switched systems with time-varying delay', *Automatica*, 2009, **45**, (1), pp. 186–193
8 Yue, D., Han, Q.: 'Network-based robust H_∞ filtering for uncertain linear systems', *IEEE Trans. Signal Process.*, 2006, **54**, (11), pp. 4293–4301
9 Bishop, A.N., Savkin, A.V., Pathirana, P.N.: 'Vision-based target tracking and surveillance with robust set-valued state estimation', *IEEE Signal Process. Lett.*, 2010, **17**, (3), pp. 289–292
10 Pathirana, P.N., Ekanayake, S.W., Savkin, A.V.: 'Fusion based 3D tracking of mobile transmitters via robust set-valued state estimation with RSS measurements', *IEEE Commun. Lett.*, 2011, **15**, (5), pp. 554–556
11 Cheng, T.M., Malyavej, V., Savkin, A.V.: 'Decentralized robust set-valued state estimation in networked multiple sensor systems', *Comput. Math. Appl.*, 2010, **59**, (8), pp. 2636–2646
12 Goncalves, E.N., Palhares, R.M., Takahashi, R.H.C.: 'H_2/H_∞ filter design for systems with ploytope-bounded uncertainty', *IEEE Trans. Signal Process.*, 2006, **54**, pp. 3620–3626
13 Xie, L., Lu, L., Zhang, D., Zhang, H.: 'Improved robust H_2/H_∞ filtering for uncertain discrete-time systems', *Automatica*, 2004, **40**, (5), pp. 873–880

14 Xiong, K., Zhang, H.Y., Liu, L.D.: 'Adaptive robust extended Kalman filter for nonlinear stochastic systems', *IET Control Theory Appl.*, 2008, **2**, pp. 239–250

15 Xiong, K., Liu, L.D., Liu, Y.W.: 'Nonlinear robust filter design for satellite attitude determination', *IET Control Theory Appl.*, 2010, **4**, (7), pp. 1222–1234

16 Ishihara, J.Y., Terra, M.H., Campos, J.C.T.: 'Robust Kalman filter for descriptor systems', *IEEE Trans. Autom. Control.*, 2006, **51**, pp. 1354–1358

17 Lu, X., Xie, L., Zhang, H., Wang, W.: 'Robust Kalman filtering for discrete-time systems with measurement delay', *IEEE Trans. Circuits Syst. II, Express Briefs*, 2007, **54**, pp. 522–526

18 Wang, J.Q., He, Z.M., Zhou, H.Y., Jiao, Y.Y.: 'Regularized robust filter for attitude determination system with relative installation error of star trackers', *Acta Astron.*, 2013, **87**, pp. 88–95

19 Tang, X.J., Liu, Z.B., Zhang, J.S.: 'Square-root quaternion cubature Kalman filtering for spacecraft attitude estimation', *Acta Astron.*, 2012, **76**, pp. 84–94

20 Xiong, K., Liu, L.D., Liu, Y.W.: 'Robust extended Kalman filtering for nonlinear systems with multiplicative noises', *Optim. Control Appl. Methods*, 2011, **32**, pp. 47–63

21 Wang, J.Q., Jiao, Y.Y., Zhou, H.Y.: 'An iterative filter for nonlinear satellite attitude determination system with multiplicative stochastic matrix', *Acta Electron. Sin.*, 2011, **39**, (6), pp. 1417–1422

22 Markley, F.L.: 'Attitude error representations for Kalman filtering', *J. Guid. Control Dyn.*, 2003, **26**, (2), pp. 311–317

23 Seo, J., Yu, M., Park, C.G. *et al.*: 'An extended robust H_∞ filter for nonlinear constrained uncertain time-varying parameters', *IEEE Trans. Signal Process.*, 2006, **54**, (6), pp. 2110–2118

24 Calafiore, G.: 'Reliable localization using set-valued nonlinear filters', *IEEE Trans. Syst. Man Cybern. Part A, Syst. Humans*, 2005, **35**, (2), pp. 189–197

25 Xie, L., Lu, L., Liu, L.: 'Robust Kalman filtering for uncertain discrete-time systems', *IEEE Trans. Autom. Control*, 1994, **39**, (6), pp. 1310–1314

26 Horn, R.A., Johnson, C.R.: 'Topic in matrix analysis' (Cambridge University Press, New York, 1991)

27 Hu, J., Wang, Z.D., Gao, H.J. *et al.*: 'Extended Kalman filtering with stochastic nonlinearities and multiple missing measurements', *Automatica*, 2012, **48**, pp. 2007–2015

28 Hu, J., Wang, Z.D., Shen, B. *et al.*: 'Quantised recursive filtering for a class of nonlinear systems with multiplicative noises and missing measurements', *Int. J. Control*, 2013, **86**, (4), pp. 650–663

29 Qian, H.M., Huang, W., Liu, B.: 'Finite-horizon robust Kalman filter for uncertain attitude estimation system with star sensor measurement delays', *Abs. Appl. Anal.*, 2014, Article ID 494060, p. 11

30 Reif, K., Unbehauen, R.: 'Stochastic stability of the discrete-time extended Kalman filter', *IEEE Trans. Autom. Control*, 1999, **44**, pp. 714–728

doi: 10.1049/iet-cta.2014.0293