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Discrete-Time Average-Consensus under Switching Network Topologies

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***Abstract*— This paper develops a distributed algorithm for average-consensus in a discrete-time framework based on a formal matrix limit definition of average-consensus. Using this algorithm, the average-consensus problem is solved under switching network topologies provided that the network switch between instantaneously balanced, connected-over-time networks. In other words, if at each instant the network is balanced and the union of graphs over every interval T is connected, then average-consensus can be achieved. An interesting product of this analysis is the notion of "deadbeat" consensus where a system of agents achieves consensus (average or otherwise) in finite time rather than asymptotically.**

I. Introduction

For teams of autonomous agents, the ability to cooperate in a decentralized manner can enhance the overall effectiveness of the team. Central to decentralized cooperation is the consensus problem which has been investigated recently by a number of researchers [1], [2], [3], [4].

In general terms, the consensus problem for a group of agents is to ensure that as time progresses each agent approaches a consistent understanding of their shared information. Average-consensus problems add the restriction of requiring that the final value (the group decision) be the exact average of the agents' initial values. Recently average-consensus has been used as a basis for distributed Kalman filters [5], [6].

In [3] Olfati-Saber and Murray propose a distributed, linear, continuous-time protocol that ensures that average-consensus is achieved asymptotically if the interaction networks connecting the agents switch between balanced, strongly connected graphs. This paper will extend those results to the discrete-time domain as well as relax the restriction of requiring the interaction topology to be strongly connected at each instant. Our main result will be to show that if the interaction topology at any instant is balanced and the *union* of the network graph is strongly connected over every interval T, then average-consensus is still achieved asymptotically. Thus, a network may at no instant be strongly connected, yet agents in a team can still achieve average-consensus.

This paper is outlined as follows. In Section II we introduce a formal definition of average-consensus as well as notation that relates the communication topology to consensus protocols. Our main results are presented in Section III. Section IV investigates the practical issues in forming an average-consensus protocol in the discrete-time framework and proposes two such protocols. The notion of *deadbeat* consensus is introduced in Section V and Section VI investigates trade-offs between asymptotic and finite-time average-consensus protocols. Finally, conclusions are offered in Section VII.

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II. Definitions and Terminology

The information flow topology between agents on a team is most naturally represented as a directed graph. For this reason, we introduce graph theoretic terminology similar to [7].

Let $A = [a_{ij}]$ be an $n \times n$ nonnegative matrix. The underlying directed graph G associated with A has vertex set $V(G) = \{1, \dots, n\}$ and a directed edge (i, j) from node i to j if and only if $a_{ji} \neq 0$ (note: some authors use the transpose of A, i.e. there is a directed edge (i, j) from i to j if and only if $a_{ij} \neq 0$). As we have defined the relationship between a matrix and its underlying graph, the nodes sending information to node i can be determined by the nonzero entries in row i. Nonzero entries in column i indicate which nodes are receiving information from node i. Note that two matrices with nonzero entries in the same locations have the same underlying graph. The neighbors, N_i, of node i are all nodes that communicate to i, i.e. $N_i = \{j \mid a_{ij} \neq 0\}$. By convention, we assume that each node can communicate with itself, so $a_{ii} > 0\ \forall i$ and $i \in N_i$.

The graphs associated with matrices can be connected in a variety of ways. Connectivity of the network can be roughly classified as follows:

- *Fully Connected:* Each node has as its neighbors all other nodes in the network.
- *Strongly Connected:* Each node has a path that follows the directed edges of the graph to every other node in the network. A direct connection to all other nodes is not necessary, but information flow from each node must reach all other nodes.
- *Spanning Tree:* At least one node has a path that follows the directed edges of the graph to every other node in the network.

Graphs can also be connected over time by considering the union of the communication links over an interval of time (i.e. the union contains all edges that were active during that interval). A reversed graph is simply a graph with the direction of the links reversed. Note that a reversed graph is associated with the transpose of the original matrix.

Each node has an associated value $x_i \in \mathbb{R}$ which represents the information on which the team must come to agreement. The set of nodes $\{1, \dots, n\}$ is said to be in consensus if $x_i = x_j$ for all i, j. When each $x_i = \frac{1}{n}\sum_j x_j[0]$

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the team is said to have reached average-consensus. A consensus protocol defines how a node should update is value of x_i based on the values of its neighbors. The simplest scheme is to require that each node update its value x_i to some weighted linear combination of its neighbors values.

$$x_i[k+1] = \sum_{j \in N_i} a_{ij} x_j[k]$$

The dynamics of the information vector $x = \{x_1, \ldots, x_n\}$ can then be defined as

$$x[k+1] = A[k]x[k]$$

where the *sign* of each entry in $A[k]$ is given by the communication topology at time k, but the value a_{ij} for the nonzero elements is determined by the protocol.

Let $\Phi_A(k, k_0) = A[k]A[k-1] \cdots A[k_0]$, then at each k the information vector can be described by

$$x[k+1] = \Phi_A(k, 0)x[0].$$

Consensus is said to be reached asymptotically if

$$\lim_{k \to \infty} \Phi_A(k, 0) = \mathbf{1}y^T \tag{1}$$

where $\mathbf{1}$ is the vector of all ones, $y_i \geq 0$, and $\mathbf{1}^T y = 1$. Notice that if Eq. (1) is satisfied, then $x \to \mathbf{1}y^T x[0]$ implying that each x_i approaches the same convex combination of the agents' initial values. Equivalently, average-consensus is said to be reached asymptotically if

$$\lim_{k \to \infty} \Phi_A(k, 0) = \frac{1}{n}\mathbf{1}\mathbf{1}^T. \tag{2}$$

III. Average-Consensus under Switching Topologies

The results for linear consensus protocols under switching interaction topologies have been well studied [2], [4] with the main result being that the union of the interaction graphs over every interval T must contain a spanning tree to reach consensus. We will draw similar conclusions with respect to average-consensus. Theorem 1 develops the conditions for each $A[k]$ that allows Eq. (2) to be satisfied. This requires the following two Lemmata.

Lemma 1 (Proposition 1 in [4]): Let $x[k+1] = A[k]x[k]$ where $A[k] = [a_{ij} \geq 0]$, $\sum_j a_{ij} = 1$, $a_{ii} > 0$ for all k, and each nonzero entry a_{ij} is both uniformly upper and lower bounded. If there exists $T \geq 0$ such that for every interval $[k, k+T]$ the union of the interaction graph across the interval contains a spanning tree, then consensus is asymptotically achieved (i.e. Eq. (1) is satisfied).

A similar result is implicit in [8]. Notice that each node has the ability to choose the weight associated with the information from each of its neighbors to ensure that its row sums to one. If the team is connected often enough (i.e. has a spanning tree over every interval of length T), then Lemma 1 ensures that consensus is reached.

Lemma 1 requires that the row sums of $A[k]$ be one and that a spanning tree be achieved in every interval of length T for consensus to be reached. Now consider the reversed dynamics $x[k+1] = B[k]x[k]$ where each *column* sum is equal to one.

Lemma 2: Let $x[k+1] = B[k]x[k]$ where $B[k] = [b_{ij} \geq 0]$, $\sum_i b_{ij} = 1$, $b_{ii} > 0$, and each nonzero entry b_{ij} is both uniformly upper and lower bounded. Under switching interaction topologies, if there exists $T \geq 0$ such that for every interval $[k, k+T]$ the union of the *reverse* interaction graph across the interval contains a spanning tree, then

$$\lim_{k \to \infty} \Phi_B(k, k_0) = y\mathbf{1}^T$$

where $y_i \geq 0$ and $y^T\mathbf{1} = 1$.

Proof: If the column sums of $B[k]$ are equal to one and a spanning tree is achieved in the reverse graph, then $B^T[k]$ has row sums of one and a spanning tree is achieved in the regular graph. By application of Lemma 1 $\lim_{k\to\infty} \Phi_{B^T}(k, k_0) = \mathbf{1}z^T$, so

$$\begin{aligned}
\lim_{k\to\infty} \Phi_{B^T}(k_0, k) &= \mathbf{1}y^T \\
[\lim_{k\to\infty} \Phi_{B^T}(k_0, k)]^T &= [\mathbf{1}y^T]^T \\
\lim_{k\to\infty} \Phi_{B^T}(k_0, k)^T &= y\mathbf{1}^T \\
\lim_{k\to\infty} \Phi_B(k, k_0) &= y\mathbf{1}^T
\end{aligned}$$

The fact that $\Phi_{B^T}(k, k_0) = \mathbf{1}z^T \Rightarrow \Phi_{B^T}(k_0, k) = \mathbf{1}y^T$ can be seen by noting that each $B^T[k]$ is row stochastic with positive diagonal entries and if the product $B^T[k]B^T[k+1] \cdots B^T[k+T]$ contains a spanning tree, then the product $B^T[k+T]B^T[k+T-1] \cdots B^T[k]$ also contains a spanning tree. Wolfowitz [9] showed that infinite products of SIA matrices (a superset of matrices that have a spanning tree and are row stochastic with positive diagonal entries) converge to the form $\mathbf{1}y^T$ in *any* product order (however, the value of y will be dependent on the actual order). ■

Theorem 1: Let $x[k+1] = A[k]x[k]$ where $A[k] = [a_{ij} \geq 0]$, $\sum_i a_{ij} = 1$, $\sum_j a_{ij} = 1$, $a_{ii} > 0$, and each nonzero entry a_{ij} is both uniformly upper and lower bounded. Under switching interaction topologies, if there exists $T \geq 0$ such that for every interval $[k, k+T]$ the union of the interaction graph across the interval is strongly connected, then Eq. (2) is satisfied and average-consensus is reached asymptotically.

Proof: Since $A[k]$ is strongly connected over each interval $[k, k+T]$, then $A[k]$ has a spanning tree in both the regular graph and the reverse graph. Therefore, the matrix $A[k]$ satisfies all the conditions in Lemma 1 *and* Lemma 2. Consequently,

$$\lim_{k \to \infty} \Phi_A(k, k_0) = \mathbf{1}y^T$$

and

$$\lim_{k \to \infty} \Phi_A(k, k_0) = z\mathbf{1}^T$$

so

$$\lim_{k \to \infty} \Phi_A(k, k_0) = \frac{1}{n}\mathbf{1}\mathbf{1}^T.$$

■

IV. Distributed Protocol

Careful examination of Theorem 1 will reveal that finding a distributed protocol to satisfy the hypotheses of the theorem will be difficult. Specifically, at each instant in time, the row *and* column sums must be equal to one. In the general consensus problem, only the row sums are required to be one. Since the neighbors of agent i are determined completely by row i, then each agent simply chooses appropriate weights for each of its neighbors values ensuring that the weights sum to one. In the average-consensus case, not only do the weights associated with the neighbors of i need to sum to one, but all nodes for which i is a neighbor must weight the information from i such that the column sum is equal to one. This section will investigate this subtlety and propose two protocols that achieve average-consensus in a distributed manner.

To illustrate the difficulty consider the network topology shown in Fig. 1.

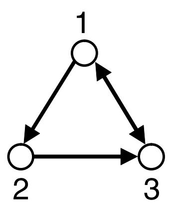

Fig. 1. Simple network over which the average-consensus problem can be solved, but which requires global information to be available.

The matrix

$$A = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

has as its underlying graph the topology shown in Fig. 1 and has row and column sums equal to one. If the underlying network topology remains fixed, then by Theorem 1, the system will achieve average-consensus asymptotically. In one sense, the protocol is distributed since each agent only uses the information received from its neighbors; however, the first agent weights all neighbors' values equally, the second agent weights its own value twice as much as its neighbors, and the third agent weights its neighbors values twice as much as its own. In order to determine the entries in A, some global knowledge of the network topology is required - i.e. there is no simple rule that an agent can use to determine the weight it gives to information from its neighbors without knowledge of the global topology.

An ideal protocol would be able to achieve average-consensus without using global information. We will investigate two protocols that impose additional restrictions on the types of graphs involved, but that achieve average-consensus without resorting to global information. The first is proposed in [3] and requires the definition of the graph Laplacian. Let L be defined element-wise as

$$\ell_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{n} \alpha_{ik}, & j = i \\ -\alpha_{ij}, & j \neq i \end{cases}$$

where $\alpha_{ij} = 1$ if there is a communication link from node j to node i and $\alpha_{ij} = 0$ otherwise (here $\mathcal{A} = [\alpha_{ij}]$ is simply the adjacency matrix of a graph G). The protocol is then defined in terms of the Laplacian

$$x[k+1] = (I - \epsilon L)x[k] \tag{3}$$

where $\epsilon \in (0, 1/\max_i \ell_{ii})$. Notice that the row sums of L are all zero by construction, so the row sums of $A = I - \epsilon L$ are all one. If $\epsilon \in (0, 1/\max_i \ell_{ii})$, then A will also be nonnegative and consensus will be guaranteed if the graph contains a spanning tree in every interval T.

Olfati-Saber and Murray show in [3] that when a graph is *balanced* then the column sums of L are zero. A balanced graph is one in which at each node the out-degree equals the in-degree, i.e. each node sends information to as many as send information to it. Notice that when G is balanced then L has column sums of zero, and A has column sums of one. So, by Theorem 1, the protocol (3) will achieve average-consensus if the network switches between instantaneously balanced networks which are strongly connected over every interval T.

Protocol (3) is almost completely distributed since each node determines the weight to associate with information from its neighbors without knowledge of the graph topology; however, all nodes must have the same value of ϵ whose upper limit is determined by the connectivity of the graph. Certainly, for fixed number of agents n, $\epsilon \in (0, 1/n]$ will ensure that A remain nonnegative and Theorem 1 will apply. This requires *a priori* knowledge of the size of the team, especially since the larger the value of ϵ the faster the rate of convergence (the second eigenvalue will be closer to zero, see [3]). Setting $\epsilon = 1/N$ where N is an upper bound on the number of agents on the team will ensure that average-consensus is achieved asymptotically.

To illustrate the applicability of this protocol consider a scenario where the network topology switches randomly from between the graphs in Fig. 2. The convergence for two values of ϵ are shown in Figures 3 and 4. In this example, the sum of the initial conditions is one. Notice that with both values of ϵ the value to which the system converges is $\frac{1}{4}$, but the larger value of ϵ gives faster convergence.

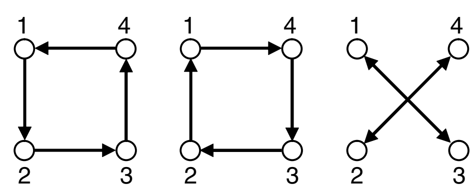

Fig. 2. Example scenario where the network topology switches randomly between these three graphs.

When every communication link is bi-directional (i.e. G is an undirected graph), then the graph is trivially balanced. In this case, it is possible to develop a protocol that can be implemented without *a priori* knowledge of team size. Assume that agents have the ability to negotiate with each

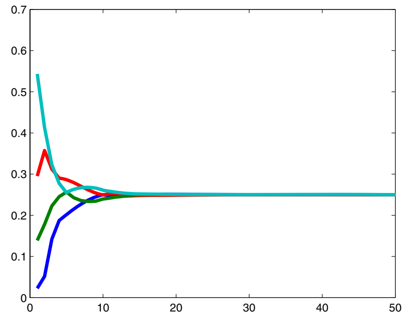

Fig. 3. Protocol (3) with $\epsilon = \frac{1}{4}$ under switching topologies.

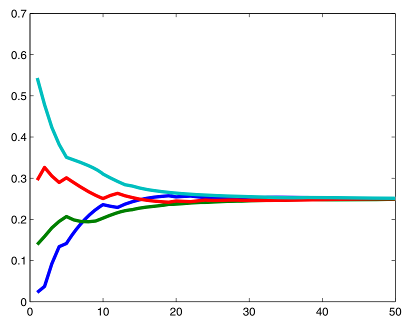

Fig. 4. Protocol (3) with $\epsilon = \frac{1}{8}$ under switching topologies.

of their neighbors to isolate the exchange of information to just one neighbor at a time. During this communication event, both agents update their values to be the exact average of the values present. For a two-agent communication event (i, j), the protocol matrix A will be the identity matrix with the exception of $a_{ij} = a_{ji} = a_{ii} = a_{jj} = \frac{1}{2}$. Notice that each $A[k]$ retains the characteristic of having row and column sums equal to one. Essentially, each agent cycles through available communication channels to isolate a single neighbor at a time and effectively change its in-degree and out-degree to one at each instant. If over every interval T the union of these simple graphs is connected, then the conditions in Theorem 1 are satisfied and average-consensus is achieved asymptotically.

An example of this protocol is shown in Fig. 6. In this scenario, the agents are connected in a static graph of the form $\{1 \leftrightarrow 2, 2 \leftrightarrow 3, 3 \leftrightarrow 4\}$. The agents negotiate with their neighbors so that each agent only communicates with one other agent at a time. For simulation purposes, this can be modeled as the system switching randomly between the graphs in Fig. 5. Observe that the final value is the exact average of the agents' initial conditions.

To summarize, the Laplacian protocol of Eq. (3) can achieve average-consensus if the interaction topology is balanced at each instant and is strongly connected over every interval T. It requires that some knowledge of the maximum connectedness or maximum number of agents be available *a priori* to determine the parameter ϵ. A second protocol

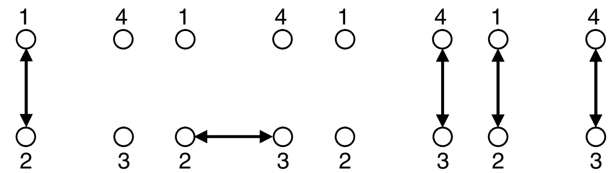

Fig. 5. Example scenario where the topology remains fixed (a path), but the agents negotiate through one of the above graphs at each instant.

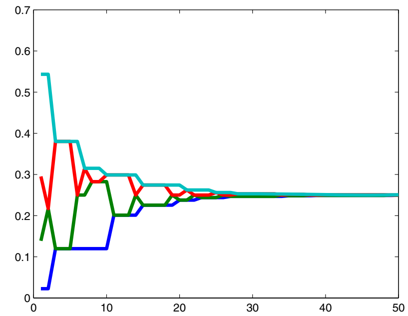

Fig. 6. Results of simple two-agent events.

reduces available communication to allow only simple two-agent interactions. This requires bi-directional communication between agents (more restrictive than balanced) but assumes no *a priori* knowledge of the network topologies or team size.

V. Deadbeat Consensus

Discrete-time systems can exhibit finite-time convergence when the poles of the system are all at zero - i.e. when a system

$$x[k+1] = Ax[k]$$

has nilpotent matrix A, then $x[k] = 0 \ \forall k > n$, regardless of initial conditions. This notion of "deadbeat" response motivates a similar investigation of consensus systems. This section will consider the conditions under which consensus can be reached in finite time.

Let $\mathcal{P}$ be a consensus protocol; specifically, let the interaction matrices generated by $\mathcal{P}$ have row sums equal to one. Note that if $\mathcal{P}$ solves either the general consensus problem or the average-consensus problem, the row sums of the interaction matrices will be one. At any time k, the value of the system given the initial conditions at $k = 0$ is

$$x[k+1] = (A[k]A[k-1] \cdots A[1]A[0])x[0].$$

Theorem 2: Let $\mathcal{P}$ be a consensus protocol. If at some instant, ℓ, the interaction topology is a fully connected graph and $\mathcal{P}$ yields the interaction matrix at that instant

$$A[\ell] = \frac{1}{n}\mathbf{1}\mathbf{1}^T$$

then the team will be exactly in consensus for all $k > \ell$.

Proof: A group of agents is in consensus if $x_i = x_j$ for every pair (i, j). Since

$$x[\ell+1] = A[\ell]x[\ell] = \frac{1}{n}\mathbf{1}\mathbf{1}^T x[\ell]$$

then each element i of $x[\ell+1]$ is

$$x_i[\ell+1] = \frac{1}{n}\sum_{j=1}^{n} x_j[\ell].$$

Therefore, the group has reached consensus at time $(\ell+1)$. Because $\mathcal{P}$ ensures that each interaction matrix after time ℓ has row sums equal to one, then for all $k > \ell$ the group remains in consensus (each node updates to a weighted sum of the same value).

To show that the same conditions on A lead to deadbeat average-consensus, notice that $\mathcal{P}$ has row and column sums equal to one at each instant so

$$\sum_{i=1}^{n} x_i[k] = \sum_{i=1}^{n} x_i[0]$$

for all $k \geq 0$. Recall that if a matrix A has row and column sums equal to one, then $\mathbf{1}$ is both a left and right eigenvector associated with eigenvalue 1, so

$$\begin{aligned}\sum_{i=1}^{n} x_i[k] &= \mathbf{1}^T x[k] \\ &= \mathbf{1}^T A[k]x[k-1] \\ &= \mathbf{1}^T x[k-1] \\ &\vdots \\ &= \mathbf{1}^T x[0] \\ &= \sum_{i=1}^{n} x_i[0].\end{aligned}$$

Therefore, for each agent i at time ℓ

$$x_i[\ell+1] = \frac{1}{n}\sum_{j=1}^{n} x_j[\ell] = \frac{1}{n}\sum_{j=1}^{n} x_j[0].$$

which implies that the group has reached average-consensus. ■

Theorem 2 requires that at some instant the communication graph is fully connected, i.e every agent can communicate with every other agent *and* that the interaction matrix generated by the consensus protocol yields $A = \frac{1}{n}\mathbf{1}\mathbf{1}^T$. One consequence of this is that deadbeat average-consensus is much more difficult to achieve than regular deadbeat consensus. This is due to the fact that average-consensus protocols do not generally yield the proper interaction matrix when the communication graph is fully connected.

The reader will note that even when a graph is fully connected neither protocol from Section IV will yield the proper interaction matrix to achieve deadbeat consensus. A consensus protocol of the form

$$x_i[k+1] = \frac{1}{|N_i|}\sum_{j \in N_i} x_j[k] \tag{4}$$

will allow regular deadbeat consensus since whenever the network is fully connected, each agent updates its value to the average of all the other agents. Unfortunately, such a simple protocol does not lead to average-consensus in the general case. Consider the balanced network shown in Fig. 7 with interaction matrix

$$A = \frac{1}{6}\begin{bmatrix} 3 & 0 & 3 & 0 \\ 2 & 2 & 0 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 0 & 3 \end{bmatrix}$$

which does not have column sums equal to one. In fact

$$\lim_{k\to\infty} A^k = \frac{1}{9}\begin{bmatrix} 2 & 3 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 3 & 2 & 2 \end{bmatrix}$$

which shows that the protocol defined by Eq. (4) is not an average-consensus protocol. It is interesting to note that if the network topologies switch between balanced *regular* graphs (graphs where all nodes that have any adjacent edge have the same degree), then this protocol does achieve average-consensus (in the general and deadbeat case).

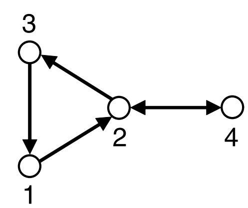

Fig. 7. Balanced graph for which a simple averaging protocol does not achieve average-consensus.

A. Example Application

Consider a fixed perimeter which is to be monitored by a team of N agents in a distributed manner (as in [10]). Let the consensus variable in the system be the size of the segment an agent is to monitor. The initial state of the system is when the first agent reaches the endpoint of the perimeter and initializes its consensus variable to the length of the perimeter. We desire average-consensus so that asymptotically, each agent monitors an equal part of the perimeter.

Using the protocol of Eq. (4) and noticing that the system will only switch between balanced regular graphs (since agents meet along a line) deadbeat average-consensus may be reached. Figure 8 shows a scenario where at no time are all agents in communication, so average-consensus is achieved asymptotically. In contrast, Figure 9 shows a scenario where agents are launched in close proximity and meet in a fully connected group near the beginning of the mission achieving deadbeat average-consensus.

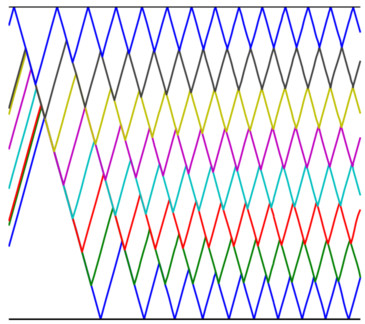

Fig. 8. Perimeter surveillance using average-consensus to distribute the team evenly along the perimeter.

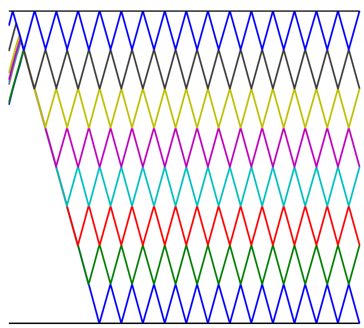

Fig. 9. Perimeter surveillance where deadbeat average-consensus occurs.

VI. Finite-Time Average-Consensus

An average-consensus protocol will invariably require a strongly connected network since each agent must be able to influence the group decision to reflect its initial condition. Obviously, if each agent transmits its initial condition to its neighbors and passes any communication received from others along, then if a strongly connected network is available, eventually all agents will have the complete set of initial conditions from which the average can be computed (clearly, this is not novel; Lynch [11] classifies such an algorithm as trivial). Using this method, all agents will have the information necessary to be in average-consensus after d steps where d is the diameter of the graph. Indeed, it may seem that the restriction to a strongly connected network eliminates any need for an asymptotic protocol. This section will investigate the trade-offs between an asymptotic protocol (such as (3)) and the message passing protocol described above.

The main advantage of an asymptotic consensus protocol is the small amount of bandwidth required - each agent needs only to send its current value. Additionally, there is no need to identify individual agents or know the number of agents in the team. On the other hand, a message passing protocol could keep track of which initial conditions it has sent to each of its neighbors and effectively limit its bandwidth to be the same as the asymptotic protocol, relying instead on repeated interaction to transmit all initial conditions. At each instant, a node's value would be the sample average, i.e. the average of all initial conditions received so far. For large networks, however, the amount of overhead and the complexity may be prohibitive. Perhaps the main advantage of the message passing protocol is the ability to utilize any type of data (not simply continuous real numbers) in any functional way, i.e. agents are not limited to average-consensus but they can come to agreement on any function of the initial conditions.

The message passing protocol effectively emulates a fully connected graph at $k = 0$ by transmitting the required information incrementally. The deadbeat nature of the protocol makes it attractive, especially when speed of convergence is an issue.

An asymptotic protocol will be useful in very large networks and in situations when the value at each node is driven by an external source (agent values are dynamic rather than static) such as distributed Kalman filtering [5]. In many cases, however, a simple message passing scheme may be more attractive due to its deadbeat nature and its ability to handle any data type.

VII. Conclusions

This paper has extended the average-consensus results of [3] to allow for networks to switch between instantaneously balanced networks that are strongly connected over every interval T. The discrete-time case has been dealt with explicitly and two asymptotic protocols presented that achieve average-consensus under switching topologies. The notion of deadbeat consensus was investigated with conclusion that general consensus problems may best be solved using a message passing mechanism rather than defining dynamics of the information variable if a strongly connected network can be assumed.

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